Multiplayer Homicidal Chauffeur Reach-Avoid Games: A Pursuit Enclosure Function Approach

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Abstract

This paper presents a multiplayer Homicidal Chauffeur reach-avoid differential game, which involves Dubins-car pursuers and simple-motion evaders. The goal of the pursuers is to cooperatively protect a planar convex region from the evaders, who strive to reach the region. We propose a cooperative strategy for the pursuers based on subgames for multiple pursuers against one evader and optimal task allocation. We introduce pursuit enclosure functions (PEFs) and propose a new enclosure region pursuit (ERP) winning approach that supports the forward analysis for the strategy synthesis in the subgames. We show that if a pursuit coalition is able to defend the region against an evader under the ERP winning, then no more than two pursuers in the coalition are necessarily needed. We also propose a steer-to-ERP approach to certify the ERP winning and synthesize the ERP winning strategy. To implement the strategy, we introduce a positional PEF and provide the necessary parameters, states, and strategies that ensure the ERP winning for both one pursuer and two pursuers against one evader. Additionally, we formulate a binary integer program using the subgame outcomes to maximize the captured evaders in the ERP winning for the pursuit task allocation. Finally, we propose a multiplayer receding-horizon strategy where the ERP winnings are checked in each horizon, the task is allocated, and the strategies of the pursuers are determined. Numerical examples are provided to illustrate the results.

Key words: Reach-avoid games; differential games; Homicidal Chauffeur; multiplayer systems; cooperative strategies

1 Introduction

Problem motivation and description: Multi-robot systems with adversarial goals, failures or improper uses, could pose significant threat to safety-critical infrastructure, such as airports and military facilities. We consider a planar multiplayer Homicidal Chauffeur reach-avoid differential game. In this game, multiple Dubins-car pursuers are used to protect a convex region against a number of malicious simple-motion evaders. To capture the most number of evaders, we propose a receding-horizon strategy for the pursuers based on subgames for multiple pursuers against one evader and optimal task allocation.

Literature review: The classical Homicidal Chauffeur differential game first proposed by Isaacs (1965), is an attractive pursuit-evasion game between a Dubins-car pursuer and a simple-motion evader, and has a long research history (Patsko & Turova, 2011). The strategies for the players in this game are quite complex mainly due to the non-linearity of the Dubins-car dynamics, and it took a long time to finally obtain its complete solution by Merz (1971). Since then, many interesting variants have been proposed, including surveillance-evasion objectives (Lewin & Breakwell, 1975; Lewin & Olsder, 1979), the

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stochastic version (Pachter & Yavin, 1981) and suicidal pedestrians (Exarchos et al., 2015). Numerical investigation and multiplayer extensions can be found in (Bopardikar et al., 2009; Falcone, 2006; Mitchell, 2002).

Recently, reach-avoid differential games (Margellos & Lygeros, 2011; Yan et al., 2023a; Zhou et al., 2012), also known as two-target differential games (Cardaliaguet, 1996; Getz & Pachter, 1981), perimeter defense games (Shishika & Kumar, 2020), or target guarding problems (Fu & Liu, 2023; Mohanan et al., 2020), have received considerable research attention due to the wide safety and security applications in antagonistic multiplayer systems. These games consider a scenario where a group of pursuers (or defenders) are tasked to protect a critical region from a group of evaders (or attackers). Due to the high-dimensional continuous joint action and state spaces, as well as complex cooperations and competitions among players, the current literature either focuses on one pursuer against one evader (Cardaliaguet, 1996; Margellos & Lygeros, 2011; Mitchell et al., 2005; Mohanan et al., 2020; Zhou et al., 2012), or solves a multiplayer game suboptimally by decomposing it into many subgames with a few players and piecing them together through task allocation (Chen et al., 2017; Garcia et al., 2020b; Lee & Bakolas, 2022b; Shishika & Kumar, 2018, 2020; Yan et al., 2023a, 2022; Yan et al., 2020; Yan et al., 2021). Three common approaches have been developed to determine the game winners or compute strategies for the players. The popular Hamilton-Jacobi-Isaacs (HJI) method using level sets relies on gridding over the state space and is ideal for low-dimensional systems (Margellos & Lygeros, 2011; Mitchell, 2002; Mitchell et al., 2005; Zhou et al., 2012). The classical characteristic method (Garcia et al., 2020a; Von Moll et al., 2022) involves integrating backward from non-unique terminal conditions (capture or entry into the protected region), and may generate complicated singular surfaces when different backward trajectories meet. The geometric method employs geometric concepts, such as Voronoi diagram (Yan et al., 2019; Zhou et al., 2016), Apollonius circle (Mohanan et al., 2020; Yan et al., 2019; Yan et al., 2020), function-based evasion space (Yan et al., 2022), and dominance region (Oyler et al., 2016), for both qualitative and quantitative analysis, and have been proved powerful especially in the case of simple-motion players.

However, there are limited works on Homicidal Chauffeur reach-avoid differential games which integrate the Homicidal Chauffeur dynamics with reach-avoid competition goals. Most of the research in reach-avoid games focuses on either complex dynamical models (e.g., Dubins car) with numerical methods, or simple dynamical models with analytical methods. HJI reachability has been applied to one-pursuer-one-evader cases (Chen et al., 2019), but would suffer from high computational burden for multiplayer games. The most relevant work is Yan et al. (2023b) in which a feedback strategy is proposed with the guaranteed capture of an evader by a pursuer, provided that some conditions on initial configurations are satisfied. In Yan et al. (2023b), it mainly considers one pursuer against one evader for a protected region with infinite area and a linear boundary, and constructs pursuit strategies based on the Apollonius circle.

Contributions: We propose a receding-horizon cooperative pursuit strategy for multiplayer Homicidal Chauffeur reach-avoid differential games, with efficient computation and guaranteed winning performance. The main contributions are as follows:

(1) For the subgame with multiple pursuers (a pursuit coalition) and one evader, we introduce pursuit enclosure functions (PEFs) and then propose a new enclosure region pursuit (ERP) winning approach. The ERP winning and its strategies can be computed through the forward analysis instead of the backward reachability from the terminal conditions that generally involves solving HJI equations.
(2) We prove that under the ERP winning, if a pursuit coalition is able to defend against an evader, then at most two pursuers in the coalition are needed. This largely simplifies the pursuit strategies, as only one-pursuer and two-pursuer coalitions are needed to ensure the win.
(3) We propose a steer-to-ERP approach in which an optimization problem is solved, to generate new ERP winning states and synthesize the corresponding ERP winning strategies, based on the known ERP winning states.
(4) To implement the strategy, a positional PEF based on players' current positions is introduced. Parameters, states and strategies that can ensure the ERP winning are presented, for the cases of one pursuer and two pursuers against one evader. Finally, a multiplayer receding-horizon strategy is proposed such that in each horizon, the number of captured evaders in the ERP winning is maximized.

Paper organization: We introduce Homicidal Chauffeur reach-avoid differential games in Section 2. The ERP winning is proposed followed by a coalition reduction in Section 3. In Section 4, we propose a steer-to-ERP approach to generate new ERP winning states. Section 5 introduces a positional PEF and presents the conditions to ensure the ERP winning. In Section 6, a multiplayer strategy is proposed. Numerical results are presented in Section 7 and we conclude the paper in Section 8.

Notation: Let $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ be the set of reals, positive reals and nonnegative reals, respectively. Let $\mathbb{R}^n$ be the set of n-dimensional real column vectors and $\|\cdot\|_2$ be the Euclidean norm. All vectors in this paper are column vectors and $\boldsymbol{x}^\top$ denotes the transpose of a vector $\boldsymbol{x} \in \mathbb{R}^n$. Let $\mathbf{0}$ denote the zero vector whose dimension will be clear from the context. Denote the unit disk in $\mathbb{R}^n$ by $\mathbb{S}^n$, i.e., $\mathbb{S}^n = \{\boldsymbol{u} \in \mathbb{R}^n \mid \|\boldsymbol{u}\|_2 \leq 1\}$. The distance between two points $\boldsymbol{x} \in \mathbb{R}^2$ and $\boldsymbol{y} \in \mathbb{R}^2$ is defined

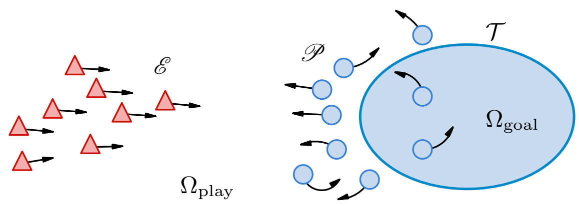

Fig. 1. Multiplayer Homicidal Chauffeur reach-avoid differential games, where a group of (red) simple-motion evaders $\mathscr{E}$, starting from a play region Ω_{play}, aim to enter a goal region Ω_{goal} protected by multiple (blue) car-model pursuers $\mathscr{P}$, and $\mathcal{T}$ is the boundary curve between Ω_{play} and Ω_{goal}.

by $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_2$. The distance between a point $\boldsymbol{x} \in \mathbb{R}^2$ and a non-empty set $\mathcal{M} \subset \mathbb{R}^2$ is defined by $d(\boldsymbol{x}, \mathcal{M}) = \inf_{\boldsymbol{y} \in \mathcal{M}} \|\boldsymbol{x} - \boldsymbol{y}\|_2$. The distance between two non-empty sets $\mathcal{M}_1$ and $\mathcal{M}_2$ is defined by $d(\mathcal{M}_1, \mathcal{M}_2) = \inf_{\boldsymbol{x} \in \mathcal{M}_1, \boldsymbol{y} \in \mathcal{M}_2} \|\boldsymbol{x} - \boldsymbol{y}\|_2$. For a vector $\boldsymbol{x} = [x, y]^\top \in \mathbb{R}^2$, let $\boldsymbol{x}^\circ = [y, -x]^\top$ be the vector obtained by rotating $\boldsymbol{x}$ in a clockwise direction by $\pi/2$. For a finite set S, we denote by $|S|$ the cardinality of S. Further notations are provided in Table 1, which will be explained in more detail later.

2 Problem Statement

2.1 Homicidal Chauffeur reach-avoid games

Consider a Homicidal Chauffeur reach-avoid differential game in an obstacle-free plane between N_p pursuers $\mathscr{P} = \{P_1, \ldots, P_{N_p}\}$ and N_e evaders $\mathscr{E} = \{E_1, \ldots, E_{N_e}\}$. Each pursuer $P_i \in \mathscr{P}$ is a Dubins car:

$$\begin{aligned}
\dot{x}_{P_i} &= v_{P_i} \cos\theta_{P_i}, \quad x_{P_i}(0) = x_{P_i}^0, \\
\dot{y}_{P_i} &= v_{P_i} \sin\theta_{P_i}, \quad y_{P_i}(0) = y_{P_i}^0, \\
\dot{\theta}_{P_i} &= v_{P_i} u_{P_i} / \kappa_i, \quad \theta_{P_i}(0) = \theta_{P_i}^0,
\end{aligned} \tag{1}$$

where $\boldsymbol{x}_{P_i} = [x_{P_i}, y_{P_i}]^\top \in \mathbb{R}^2$, $\theta_{P_i} \in \mathbb{R}$ and u_{P_i} are P_i's position, heading and control input, respectively, and $v_{P_i}, \kappa_i \in \mathbb{R}_{>0}$ are P_i's maximum speed and minimum turning radius, respectively. Assume that u_{P_i} belongs to $\mathbb{U}_P = \{u : [0, \infty) \to \mathbb{S}^1 \,|\, u \text{ is piecewise smooth}\}$. The initial position and heading are $\boldsymbol{x}_{P_i}^0 = [x_{P_i}^0, y_{P_i}^0]^\top \in \mathbb{R}^2$ and $\theta_{P_i}^0 \in \mathbb{R}$, respectively. Each evader $E_j \in \mathscr{E}$ has a simple motion:

$$\begin{aligned}
\dot{x}_{E_j} &= v_{E_j} u_{E_j}^x, \quad x_{E_j}(0) = x_{E_j}^0, \\
\dot{y}_{E_j} &= v_{E_j} u_{E_j}^y, \quad y_{E_j}(0) = y_{E_j}^0,
\end{aligned} \tag{2}$$

where $\boldsymbol{x}_{E_j} = [x_{E_j}, y_{E_j}]^\top \in \mathbb{R}^2$ and $\boldsymbol{u}_{E_j} = [u_{E_j}^x, u_{E_j}^y]^\top$ are E_j's position and control input, respectively, $v_{E_j} \in \mathbb{R}_{>0}$ is E_j's maximum speed, and $\boldsymbol{u}_{E_j}$ belongs to $\mathbb{U}_E = \{\boldsymbol{u} : [0, \infty) \to \mathbb{S}^2 \,|\, \boldsymbol{u} \text{ is piecewise smooth}\}$. The initial position is $\boldsymbol{x}_{E_j}^0 = [x_{E_j}^0, y_{E_j}^0]^\top \in \mathbb{R}^2$.

We denote the speed ratio between P_i and E_j by $\alpha_{ij} = v_{P_i}/v_{E_j}$ and consider faster pursuers, i.e., $\alpha_{ij} > 1$. The capture radius of pursuer P_i is $r_i > 0$. An evader is captured by a pursuer if the latter is pursuing the former and their Euclidean distance is less than or equal to the pursuer's capture radius. This is slightly different from the pioneering works (Merz, 1971, 1974) by Merz, where the capture occurs when their Euclidean distance is less than the capture radius for the convenience of determining the *usable part* (Isaacs, 1965) of the terminal surface. However, since the approach we propose below is based on the forward analysis with no need to work retrogressively from the terminal surface, we here consider a closed capture condition such that the case when the distance is equal to the capture radius is also included.

The plane $\mathbb{R}^2$ is split by a closed convex curve $\mathcal{T}$ called the *target curve*, into a *goal region* Ω_{goal} and a *play region* Ω_{play} as illustrated in Fig. 1, and formally, they are respectively defined by

$$\begin{aligned}
\Omega_{\text{goal}} &= \{\boldsymbol{y} \in \mathbb{R}^2 \,|\, g(\boldsymbol{y}) \le 0\}, \\
\mathcal{T} &= \{\boldsymbol{y} \in \mathbb{R}^2 \,|\, g(\boldsymbol{y}) = 0\}, \\
\Omega_{\text{play}} &= \{\boldsymbol{y} \in \mathbb{R}^2 \,|\, g(\boldsymbol{y}) > 0\},
\end{aligned}$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ is a twice differentiable function, i.e., $g_{\boldsymbol{y}} \triangleq \frac{\mathrm{d}g}{\mathrm{d}\boldsymbol{y}}$ and $g_{\boldsymbol{yy}} \triangleq \frac{\mathrm{d}^2 g}{\mathrm{d}\boldsymbol{y}^2}$ exist, such that Ω_{goal} is nonempty, compact and convex. The evasion team $\mathscr{E}$ aims to send as many evaders initially in the play region as possible into the goal region before being captured by the pursuit team $\mathscr{P}$ who guards the goal region.

2.2 Information structure

The information available to each player plays an important role in determining game outcomes (Cardaliaguet, 1996; Elliott & Kalton, 1972; Isaacs, 1965; Mitchell et al., 2005). Since this paper aims to propose strategies for the pursuit team $\mathscr{P}$, we will build the information structure from the pursuers' perspective. We adopt the same information structure used in Yan et al. (2023b). Under this information structure, the pursuit team makes decisions about its current control input with the information of all players' current positions, plus the evasion team's current control input (i.e., speeds and headings), which will be explicitly defined below, while the evasion team is assumed to have only the access to all players' current positions. The maximum speeds of all players and the information about Ω_{goal}, $\mathcal{T}$ and Ω_{play} are known by both teams.

As in Yan et al. (2023b), we discuss how to practically access an evader's current control input (i.e., speed and heading under (2)). Literature provides several methods to estimate the exogenous control signals of a moving object (such as an unmanned aerial vehicle), which can be used to estimate the evader's speed and heading in

Table 1. Notation Table

Symbol	Description	Symbol	Description
$X_{ij} = (\boldsymbol{x}_{P_i}, \theta_{P_i}, \boldsymbol{x}_{E_j})$	state of P_i and E_j	$\mathcal{S}_{ij}$	set of X_{ij} when E_j is not captured by P_i
$X_c = \{(\boldsymbol{x}_{P_i}, \theta_{P_i})\}_{i\in c}$	state of pursuit coalition P_c	$f_c = \{f_i\}_{i\in c}$	set of PEFs for pursuit coalition P_c
$X_{cj} = (X_c, \boldsymbol{x}_{E_j})$	state of P_c and E_j	$\mathcal{S}_{cj}$	set of X_{cj} when E_j is not captured by P_c
$\mathbb{E}(X_{cj}; f_c)$	enclosure region for P_c and E_j	$\varrho(X_{cj}; f_c)$	safe distance for P_c and E_j
X_{cj}^t	state after t starting from X_{cj}	$\mathcal{P}(X_{cj})$	convex program for computing $\varrho(X_{cj}; f_c)$
$f_i^{\text{ps}}, f_c^{\text{ps}}$	positional PEF(s)	$\mathcal{X}_{cj}^{\text{ps}}$	an initial set of ERP winning states
$\varrho(X_{ij}; f_i^{\text{ps}}), \varrho(X_{cj}; f_c^{\text{ps}})$	positional safe distance	$\mathcal{P}^{\text{ps}}(X_{cj})$	convex program for computing $\varrho(X_{cj}; f_c^{\text{ps}})$

our problem (Battistini & Shima, 2014; Fonod & Shima, 2018; Mohanan et al., 2020), or assumes the evader's speed and heading are accessible (Exarchos et al., 2015). Advanced sensor systems, such as phased-array radars, can provide high-accuracy measurements of the velocity (i.e., speed and heading) of a moving object (Pihl et al., 2012; Soumekh, 1997).

3 Enclosure Region Pursuit Winning and Coalition Reduction against One Evader

It is hard to deal with multiple pursuers against multiple evaders directly due to complicated cooperation among team members and the lack of sensible matching rules between the players of different teams (Chen et al., 2017; Shishika & Kumar, 2018, 2020; Yan et al., 2022). As an alternative, we split the whole game into many subgames, from which cooperative strategies and matching rules are extracted, thus comprising the team strategies. Such decomposition into subgames dates back to Li et al. (2005) where multi-player pursuit-evasion games were suboptimally solved based on the outcomes of all pairs of a pursuer and an evader, and was further generalised into a *dynamic divide and conquer* approach (Makkapati & Tsiotras, 2019). The subgames considered in this section focus on multiple pursuers and a single evader.

We first introduce the following definitions and notations specialized for multiple pursuers. For any non-empty index set $c \in 2^{\{1,\dots,N_p\}}$, let $P_c = \{P_i \in \mathscr{P} \mid i \in c\}$ be an element of $2^{\mathscr{P}}$, and we refer to P_c as a *pursuit coalition* containing pursuer P_i if $i \in c$. We denote by X_c and U_c the states (positions and headings) and control inputs of all pursuers in P_c, respectively, i.e., $X_c = \{(\boldsymbol{x}_{P_i}, \theta_{P_i})\}_{i\in c}$ and $U_c = \{u_{P_i}\}_{i\in c}$.

3.1 Enclosure region pursuit winning

Consider a subgame between a pursuit coalition P_c and an evader E_j, in which P_c wins the game if E_j can never reach Ω_{goal} before being captured, while E_j wins if it reaches Ω_{goal} prior to the capture. Our goal in this section is to determine, given the initial states, who wins the game. A complete solution to this qualitative problem involves solving an induced HJI equation numerically. However, the standard HJI method can only efficiently handle systems of up to five states (Chen et al., 2018), while this subgame has $3|c|+2$ states. Note that P_c wins the game in two possible ways: capture E_j in Ω_{play} or infinitely delay E_j's entry into Ω_{play}. We instead establish a stronger winning condition that a dynamic region containing the evader consistently stays outside the goal region before the capture. This motivates our function-induced pursuit winning and strategies below. Actually, the core reasoning behind this winning condition was implicitly utilised in Lee & Bakolas (2022a,b); Yan et al. (2022) when all players have the simple motion, where the dynamic region is Apollonius circle (Isaacs, 1965) or it generalisation. We first introduce a class of functions.

Definition 1 (Pursuit enclosure function). *For $P_i \in \mathscr{P}$ and $E_j \in \mathscr{E}$, let $X_{ij} = (\boldsymbol{x}_{P_i}, \theta_{P_i}, \boldsymbol{x}_{E_j})$ and $\mathcal{S}_{ij} = \{X_{ij} \in \mathbb{R}^2 \times [0, 2\pi) \times \mathbb{R}^2 \mid \|\boldsymbol{x}_{P_i} - \boldsymbol{x}_{E_j}\|_2 > r_i\}$. Then, a function $f : \mathbb{R}^2 \times \mathcal{S}_{ij} \to \mathbb{R}$ is a* pursuit enclosure function (PEF) *if for each $X_{ij} \in \mathcal{S}_{ij}$, it satisfies the following conditions:*

(1) $\mathbb{E}(X_{ij}; f) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid f(\boldsymbol{x}, X_{ij}) \geq 0\}$ is compact and strictly convex. We call $\mathbb{E}(X_{ij}; f)$ the enclosure region *of P_i against E_j via f;*
(2) if $f(\boldsymbol{x}, X_{ij}) = 0$, then f is differentiable in both $\boldsymbol{x}$ and X_{ij};
(3) $f(\boldsymbol{x}_{E_j}, X_{ij}) \geq 0$.

The conditions (1) and (3) imply that the evader E_j is contained in the enclosure region $\mathbb{E}(X_{ij}; f)$ (Fig. 2 illustrates four enclosure regions, each of which has a dotted boundary). We are now ready to define a new class of pursuit winning conditions based on the PEFs. Let $X_{cj} = (X_c, \boldsymbol{x}_{E_j})$ be the state of the system comprising of P_c and E_j, and $\mathcal{S}_{cj} = \{X_{cj} \mid \forall i \in c, \|\boldsymbol{x}_{P_i} - \boldsymbol{x}_{E_j}\|_2 > r_i\}$ be the set of states such that E_j is not captured by any pursuer in P_c. Unless otherwise stated we assume that $X_{cj} \in \mathcal{S}_{cj}$ hereinafter. In this subgame, a strategy of P_i ($i \in c$) under the information structure in Section 2.2 is a mapping $u_{P_i} : \mathcal{S}_{cj} \times \mathbb{S}^2 \to \mathbb{S}^1$, and a strategy of E_j is a mapping $\boldsymbol{u}_{E_j} : \mathcal{S}_{cj} \to \mathbb{S}^2$. Note that the strategy u_{P_i} requires the current control input of E_j (i.e., speed and heading), but not the strategy of E_j.

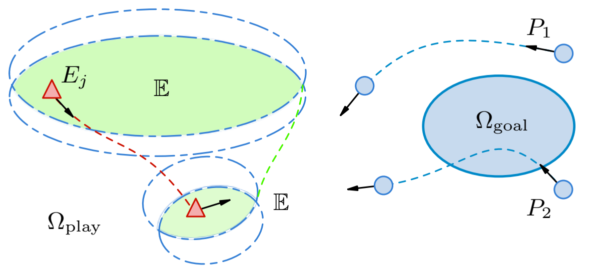

Fig. 2. If P_1 and P_2 can cooperatively ensure that the intersection $\mathbb{E}$ (green) of two enclosure regions containing E_j never intersects with Ω_{goal}, then the enclosure region pursuit (ERP) winning is achieved.

Suppose that P_c is endowed with a set f_c of PEFs, where $f_c = \{f_i\}_{i\in c}$ and f_i is a PEF for pursuer P_i ($i \in c$), and we omit the subscript j for PEFs for simplicity. Then, the intersection of the enclosure regions of P_i against E_j via f_i for all $i \in c$, is

$$\begin{aligned}\mathbb{E}(X_{cj}; f_c) &\triangleq \cap_{i\in c}\mathbb{E}(X_{ij}; f_i)\\ &= \{\boldsymbol{x} \in \mathbb{R}^2 \mid f_i(\boldsymbol{x}, X_{ij}) \geq 0, i \in c\}.\end{aligned}$$

The green region in Fig. 2 is the intersection of two enclosure regions of P_1 and P_2 against E_j. By Definition 1, $\mathbb{E}(X_{cj}; f_c)$ is nonempty because $\boldsymbol{x}_{E_j} \in \mathbb{E}(X_{cj}; f_c)$, and it is compact and strictly convex. Letting

$$\varrho(X_{cj}; f_c) \triangleq d(\mathbb{E}(X_{cj}; f_c), \Omega_{\text{goal}})$$

be the distance between the region $\mathbb{E}(X_{cj}; f_c)$ and the goal region Ω_{goal}, we next introduce *safe distance* for P_c against E_j.

Definition 2 (Safe distance). *Consider a pursuit coalition P_c and an evader E_j. Given a set f_c of PEFs, the* safe distance *of a state $X_{cj} \in \mathcal{S}_{cj}$ under f_c is $\varrho(X_{cj}; f_c)$.*

Since $\boldsymbol{x}_{E_j} \in \mathbb{E}(X_{cj}; f_c)$, then $d(\boldsymbol{x}_{E_j}, \Omega_{\text{goal}}) \geq \varrho(X_{cj}; f_c)$ for all $X_{cj} \in \mathcal{S}_{cj}$. This implies that if the safe distance is positive, i.e., $\varrho(X_{cj}; f_c) > 0$, then E_j resides outside of Ω_{goal}. Let X_{cj}^t be the system state at $t \geq 0$ starting from X_{cj} under control inputs U_c and $\boldsymbol{u}_{E_j}$. We next introduce a new pursuit winning strategy using the safe distance.

Definition 3 (ERP winning state and strategy). *Given a set f_c of PEFs, a state $X_{cj} \in \mathcal{S}_{cj}$ is an enclosure region pursuit (ERP) winning state, if there exists a pursuit strategy U_c for P_c such that the safe distance is positive from X_{cj} for all $t \geq 0$, i.e., $\varrho(X_{cj}^t; f_c) > 0$ for all $t \geq 0$, regardless of $\boldsymbol{u}_{E_j}$. Such a strategy U_c is called an ERP winning strategy.*

By Definition 3, if a state X_{cj} is an ERP winning state, then the pursuit coalition can ensure that, from this state, the evader can never enter Ω_{goal} prior to the capture, by using the ERP winning strategy, as depicted in Fig. 2. Since $\mathbb{E}(X_{cj}; f_c)$ is strictly convex and Ω_{goal} is convex, computing the safe distance $\varrho(X_{cj}; f_c)$ involves solving a convex optimization problem.

Definition 4 (Computing the safe distance). *Given a set f_c of PEFs and a state $X_{cj} \in \mathcal{S}_{cj}$, let $(\boldsymbol{x}_I, \boldsymbol{x}_G)$ be the solution of the convex optimization problem $\mathcal{P}(X_{cj})$:*

$$\begin{aligned}\underset{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^2\times\mathbb{R}^2}{\text{minimize}} \quad & d(\boldsymbol{x}, \boldsymbol{y})\\ \text{subject to} \quad & f_i(\boldsymbol{x}, X_{ij}) \geq 0,\ g(\boldsymbol{y}) \leq 0,\ \forall i \in c.\end{aligned} \tag{3}$$

Then, the safe distance is $\varrho(X_{cj}; f_c) = d(\boldsymbol{x}_I, \boldsymbol{x}_G)$.

A recent relevant work by Lee and Bakolas (Lee & Bakolas, 2022b), which focuses on simple-motion players in $\mathbb{R}^n$ with point capture, requires computing the distance between two convex sets and has proposed an interesting alternating projection algorithm to solve an induced convex optimization problem similar to (3). This alternating projection algorithm can also be used to solve (3) with a slight modification.

3.2 Coalition reduction

The convex optimization problem $\mathcal{P}(X_{cj})$ in (3) is used to compute the safe distance which plays a vital role in the following analysis. Noting this, we next prove that given X_{cj}, there are at most two PEFs in f_c which are necessary to solve $\mathcal{P}(X_{cj})$, greatly simplifying the computation. This also implies that at most two pursuers in P_c work when computing $\boldsymbol{x}_I$. Similar coalition reduction for simple-motion players can be found in Von Moll et al. (2020); Yan et al. (2022); Yan et al. (2020).

Instead of concentrating on the specific convex problem (3), we consider support constraints (defined later) for more general convex optimization problems. Consider the optimization problem

$$\mathcal{P}:\ \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}}\ \phi(\boldsymbol{x}) \quad \text{subject to} : \boldsymbol{x} \in \bigcap_{i\in\{1,\dots,m\}} \mathcal{X}_i \tag{4}$$

where $\phi(\boldsymbol{x})$ is a real-valued function in $\boldsymbol{x}$ and $\mathcal{X}_i$ ($i = 1,\dots,m$) are closed convex sets. Then, define the programs $\mathcal{P}_k$, $k = 1,\dots,m$ obtained from $\mathcal{P}$ by removing the kth constraint

$$\mathcal{P}_k:\ \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}}\ \phi(\boldsymbol{x}) \quad \text{subject to} : \boldsymbol{x} \in \bigcap_{i\in\{1,\dots,m\}\setminus k} \mathcal{X}_i.$$

We assume program $\mathcal{P}$ and the programs $\mathcal{P}_k$ admit an optimal solution, say $\boldsymbol{x}^*$ and $\boldsymbol{x}_k^*$, respectively, and let $J^* = \phi(\boldsymbol{x}^*)$ and $J_k^* = \phi(\boldsymbol{x}_k^*)$.

Definition 5 (Support constraint). *The kth constraint $\mathcal{X}_k$ is a* support constraint *for $\mathcal{P}$ if $J_k^* < J^*$.*

We first recall a well-known result due to Calafiore and Campi; see Calafiore & Campi (2006).

Lemma 1 (Calafiore and Campi). *If $\phi(\boldsymbol{x}) = \boldsymbol{c}^\top \boldsymbol{x}$ and program $\mathcal{P}$ and the programs $\mathcal{P}_k$ admit a unique optimal solution, then the number of support constraints for problem $\mathcal{P}$ is at most n.*

By Lemma 1, the convex program $\mathcal{P}(X_{cj})$ in (3) can be solved through a convex program with fewer constraints.

Theorem 1 (Constraint reduction). *Let J^*_{cj} be the optimal value of the convex program $\mathcal{P}(X_{cj})$. Then, if $J^*_{cj} > 0$, there exists a subcoalition $\bar{c}$ of c such that $|\bar{c}| \leq 2$ and $J^*_{cj} = J^*_{\bar{c}j}$, where $J^*_{\bar{c}j}$ is the optimal value of the convex optimization problem $\mathcal{P}(X_{\bar{c}j})$:*

$$\begin{aligned} \underset{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^2\times\mathbb{R}^2}{\text{minimize}} \quad & d(\boldsymbol{x},\boldsymbol{y}) \\ \text{subject to} \quad & f_i(\boldsymbol{x}, X_{ij}) \geq 0,\ g(\boldsymbol{y}) \leq 0,\ \forall i \in \bar{c}. \end{aligned} \tag{5}$$

Proof. We prove the theorem by formulating the convex program $\mathcal{P}(X_{cj})$ in (3) as a special case of (4) and then using Lemma 1 to reduce the constraints, which leads to (5). Since $J^*_{cj} > 0$, then $\varrho(X_{cj}; f_c) = d(\boldsymbol{x}_I, \boldsymbol{x}_G) > 0$ and therefore $d(\mathbb{E}(X_{cj}; f_c), \Omega_{\text{goal}}) > 0$, where $(\boldsymbol{x}_I, \boldsymbol{x}_G)$ is the optimal solution to $\mathcal{P}(X_{cj})$. Since $\mathbb{E}(X_{cj}; f_c)$ is closed and strictly convex and Ω_{goal} is closed and convex, then the optimal solution $(\boldsymbol{x}_I, \boldsymbol{x}_G)$ is unique.

Then, $\mathcal{P}(X_{cj})$ in (3) is equivalent to the following problem $\overline{\mathcal{P}}$:

$$\begin{aligned} \underset{(\boldsymbol{x},\gamma)\in\mathbb{R}^2\times\mathbb{R}}{\text{minimize}} \quad & \gamma \\ \text{subject to} \quad & f_i(\boldsymbol{x}, X_{ij}) \geq 0,\ \forall i \in c, \\ & d(\boldsymbol{x}, \boldsymbol{x}_G) - \gamma \leq 0. \end{aligned} \tag{6}$$

In (6), $(\boldsymbol{x}, \gamma)$ is three-dimensional, and there are $|c|+1$ convex constraints as the convexity of $d(\boldsymbol{x}, \boldsymbol{x}_G) - \gamma \leq 0$ is straightforward. In order to apply Lemma 1, we next show the uniqueness of the solution. We only consider the case $|c| \geq 3$, as we can take $\bar{c} = c$ when $|c| \leq 2$.

Since $\mathcal{P}(X_{cj})$ has the unique optimal solution $(\boldsymbol{x}_I, \boldsymbol{x}_G)$, then $\overline{\mathcal{P}}$ has a unique optimal solution $(\boldsymbol{x}_I, \gamma^*)$, where $\gamma^* = d(\boldsymbol{x}_I, \boldsymbol{x}_G)$. For $i \in c$, we let $\overline{\mathcal{P}}_i$ be the convex program obtained from $\overline{\mathcal{P}}$ by removing the constraint $f_i(\boldsymbol{x}, X_{ij}) \geq 0$. Then, following the same argument to $\overline{\mathcal{P}}$, we have that $\overline{\mathcal{P}}_i$ admits a unique optimal solution. We do not need to remove the constraint $d(\boldsymbol{x}, \boldsymbol{x}_G) - \gamma \leq 0$ as it is a support constraint for $\overline{\mathcal{P}}$.

By Lemma 1, we conclude that $\overline{\mathcal{P}}$ has at most 3 support constraints. Since $d(\boldsymbol{x}, \boldsymbol{x}_G) - \gamma \leq 0$ is a support constraint, then there are at most two support constraints among $f_i(\boldsymbol{x}, X_{ij}) \geq 0$ for $i \in c$. □

4 ERP Winning State Generation

If a set of states are known to be ERP winning, establishing the connection between this set and other states in $\mathcal{S}_{cj}$ can help find more ERP winning states. This section proposes a *steer-to-ERP* approach to verify whether a state is ERP winning based on the known ERP winning states, via steering the system to a known ERP winning state.

The following notations will be required. For a PEF f_i, we define $f_{i,\boldsymbol{x}}(\boldsymbol{x}, X_{ij}) = \partial f(\boldsymbol{x}, X_{ij})/\partial \boldsymbol{x}$, $f_{i,P}(\boldsymbol{x}, X_{ij}) = \partial f(\boldsymbol{x}, X_{ij})/\partial \boldsymbol{x}_{P_i}$, $f_{i,\theta}(\boldsymbol{x}, X_{ij}) = \partial f(\boldsymbol{x}, X_{ij})/\partial \theta_{P_i}$, and $f_{i,E}(\boldsymbol{x}, X_{ij}) = \partial f(\boldsymbol{x}, X_{ij})/\partial \boldsymbol{x}_{E_j}$. For two distinct points $\boldsymbol{x} \in \mathbb{R}^2$ and $\boldsymbol{y} \in \mathbb{R}^2$, let $d_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) = \partial d(\boldsymbol{x}, \boldsymbol{y})/\partial \boldsymbol{x}$ and $d_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) = \partial d(\boldsymbol{x}, \boldsymbol{y})/\partial \boldsymbol{y}$. Let $\boldsymbol{e}_{\theta_{P_i}} = [\cos\theta_{P_i}, \sin\theta_{P_i}]^\top$.

Theorem 2 (Steer-to-ERP approach). *Given a set f_c of PEFs and a state $X_{cj} \in \mathcal{S}_{cj}$, if there exists a pursuit strategy U_c^1 of P_c and a time horizon $T \in \mathbb{R}_{\geq 0}$ such that $X_{cj}^{t^\star}$ is an ERP winning state for some $t^\star \in [0, T]$ regardless of $\boldsymbol{u}_{E_j}$, and $\varrho(X_{cj}; f_c) > \max\{-VT, 0\}$, where V is the optimal value of problem*

$$\begin{aligned} \text{minimize} \quad & \sum\nolimits_{i\in c} v_{P_i}\lambda_i\Big(f_{i,P}^\top \boldsymbol{e}_{\theta_{P_i}} + \frac{|f_{i,\theta}|}{\kappa_i} + \frac{\|f_{i,E}\|_2}{\alpha_{ij}}\Big) \\ \text{variables} \quad & \boldsymbol{x}_I \in \mathbb{R}^2, \boldsymbol{x}_G \in \mathbb{R}^2, X'_{cj} \in \mathcal{S}_{cj}, \lambda_i, \lambda_g, i \in c \\ \text{subject to} \quad & \boldsymbol{0} = \lambda_g g_{\boldsymbol{y}}(\boldsymbol{x}_G) + \sum\nolimits_{i\in c} \lambda_i f_{i,\boldsymbol{x}}(\boldsymbol{x}_I, X'_{ij}) \\ & \boldsymbol{0} = d_{\boldsymbol{y}}(\boldsymbol{x}_I, \boldsymbol{x}_G) + \lambda_g g_{\boldsymbol{y}}(\boldsymbol{x}_G) \\ & f_i(\boldsymbol{x}_I, X'_{ij}) \geq 0, \lambda_i \leq 0, \lambda_i f_i(\boldsymbol{x}_I, X'_{ij}) = 0 \\ & i \in c, g(\boldsymbol{x}_G) = 0, \lambda_g \geq 0 \end{aligned} \tag{7}$$

then X_{cj} is an ERP winning state. Moreover, if U_c^2 is an ERP winning strategy for $X_{cj}^{t^\star}$, then taking U_c^1 for $[0, t^\star]$ and U_c^2 for $(t^\star, \infty)$ is an ERP winning strategy for X_{cj}.

Proof. Since X_{cj} reaches an ERP winning state $X_{cj}^{t^\star}$ for some $t^\star \in [0, T]$ under U_c^1 regardless of $\boldsymbol{u}_{E_j}$, then X_{cj} is an ERP winning sate if the positive safe distance is kept before $X_{cj}^{t^\star}$ is reached. To that end, we first compute the minimum speed of the intersection of enclosure regions moving away from Ω_{goal} and then obtain the minimum (possibly negative) increment of the safe distance during the time horizon T. Thus, X_{cj} is an ERP winning sate if the initial safe distance $\varrho(X_{cj}; f_c)$ can compensate this minimum increment.

Consider a state $X'_{cj} \in \mathcal{S}_{cj}$ such that $\varrho(X'_{cj}; f_c) > 0$, that is, $d(\mathbb{E}(X'_{cj}; f_c), \Omega_{\text{goal}}) > 0$. Since $\mathbb{E}(X'_{cj}; f_c)$ is strictly convex and Ω_{goal} is convex, there exists a unique solution $(\boldsymbol{x}_I, \boldsymbol{x}_G) \in \mathbb{E}(X'_{cj}; f_c) \times \Omega_{\text{goal}}$ to $\mathcal{P}(X'_{cj})$ in (3), and we also have $d(\boldsymbol{x}_I, \boldsymbol{x}_G) > 0$.

By *Karush-Kuhn-Tucker (KKT) conditions* for (3), the solution $(\boldsymbol{x}_I, \boldsymbol{x}_G)$ satisfies

$$\mathbf{0} = d_{\boldsymbol{x}}(\boldsymbol{x}_I, \boldsymbol{x}_G) + \sum\nolimits_{i\in c} \lambda_i f_{i,\boldsymbol{x}}(\boldsymbol{x}_I, X'_{ij}) \tag{8a}$$

$$\mathbf{0} = d_{\boldsymbol{y}}(\boldsymbol{x}_I, \boldsymbol{x}_G) + \lambda_g g_{\boldsymbol{y}}(\boldsymbol{x}_G) \tag{8b}$$

$$f_i(\boldsymbol{x}_I, X'_{ij}) \geq 0,\ \lambda_i \leq 0,\ \lambda_i f_i(\boldsymbol{x}_I, X'_{ij}) = 0,\ i \in c \tag{8c}$$

$$g(\boldsymbol{x}_G) \leq 0,\ \lambda_g g(\boldsymbol{x}_G) = 0,\ \lambda_g \geq 0 \tag{8d}$$

where X'_{ij} is a part of X'_{cj} corresponding to P_i, and λ_i and λ_g are the Lagrange multipliers. The complementary slackness condition (8c) implies that the index set c can be classified into two disjoint index sets $c^{=0}$ and $c^{>0}$ ($c^{>0}$ may be empty) where

$$\begin{cases} f_i(\boldsymbol{x}_I, X'_{ij}) = 0,\ \lambda_i \leq 0, & \text{if } i \in c^{=0}, \\ f_i(\boldsymbol{x}_I, X'_{ij}) > 0,\ \lambda_i = 0, & \text{if } i \in c^{>0}. \end{cases} \tag{9}$$

Then the region $\mathbb{E}(X'_{cj}; f_c)$ moves away from Ω_{goal} with the speed $\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X'_{cj}; f_c)$ that equals

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} d(\boldsymbol{x}_I, \boldsymbol{x}_G) = d_{\boldsymbol{x}}^\top(\boldsymbol{x}_I, \boldsymbol{x}_G)\dot{\boldsymbol{x}}_I + d_{\boldsymbol{y}}^\top(\boldsymbol{x}_I, \boldsymbol{x}_G)\dot{\boldsymbol{x}}_G \\ &= -\sum\nolimits_{i\in c} \lambda_i f_{i,\boldsymbol{x}}^\top(\boldsymbol{x}_I, X'_{ij})\dot{\boldsymbol{x}}_I - \lambda_g g_{\boldsymbol{y}}^\top(\boldsymbol{x}_G)\dot{\boldsymbol{x}}_G \\ &= -\sum\nolimits_{i\in c} \lambda_i f_{i,\boldsymbol{x}}^\top(\boldsymbol{x}_I, X'_{ij})\dot{\boldsymbol{x}}_I, \end{aligned} \tag{10}$$

where the second equality is due to (8), and the third equality follows noting that $\boldsymbol{x}_G$ is always at the boundary of Ω_{goal}, i.e., $g(\boldsymbol{x}_G) \equiv 0$ and thus $g_{\boldsymbol{y}}^\top(\boldsymbol{x}_G)\dot{\boldsymbol{x}}_G = 0$. For any $i \in c^{=0}$, since $\boldsymbol{x}_I$ is always at the boundary of $\mathbb{E}(X'_{ij}; f_i)$, then $f_i(\boldsymbol{x}_I, X'_{ij}) \equiv 0$, and according to the condition 2 in Definition 1, f_i is differentiable in $\boldsymbol{x}$, $\boldsymbol{x}_{P_i}$, θ_{P_i} and $\boldsymbol{x}_{E_j}$. Then, we have $\frac{\mathrm{d}}{\mathrm{d}t} f_i(\boldsymbol{x}_I, X'_{ij}) = 0$, implying that

$$\begin{aligned} f_{i,\boldsymbol{x}}^\top(\boldsymbol{x}_I, X'_{ij})\dot{\boldsymbol{x}}_I &= -f_{i,P}^\top\dot{\boldsymbol{x}}_{P_i} - f_{i,\theta}\dot{\theta}_{P_i} - f_{i,E}^\top\dot{\boldsymbol{x}}_{E_j} \\ &= -v_{P_i} f_{i,P}^\top \boldsymbol{e}_{\theta_{P_i}} - f_{i,\theta} v_{P_i} u_{P_i}/\kappa_i - v_{E_j} f_{i,E}^\top \boldsymbol{u}_{E_j}, \end{aligned} \tag{11}$$

where the second equality follows from (1) and (2).

Then, the minimum speed of $\mathbb{E}(X'_{cj}; f_c)$ moving away from Ω_{goal}, denoted by $V(X'_{cj})$, is equal to

$$\begin{aligned} &\min_{U_c \in \mathbb{U}_P^c} \min_{\boldsymbol{u}_{E_j} \in \mathbb{U}_E} \frac{\mathrm{d}}{\mathrm{d}t}\varrho(X'_{cj}; f_c) \\ &= \sum\nolimits_{i\in c} \lambda_i (v_{P_i} f_{i,P}^\top \boldsymbol{e}_{\theta_{P_i}} + |f_{i,\theta}| v_{P_i}/\kappa_i + v_{E_j}\|f_{i,E}\|_2), \end{aligned}$$

where (10), (11) and $\lambda_i \leq 0$ are used. The constraint (8) is written as (7) using $d_{\boldsymbol{x}}(\boldsymbol{x}_I, \boldsymbol{x}_G) = -d_{\boldsymbol{y}}(\boldsymbol{x}_I, \boldsymbol{x}_G)$. Thus, the minimum speed of the region $\mathbb{E}(X'_{cj}; f_c)$ moving away from Ω_{goal} over $\mathcal{S}_{cj}$ is $V = \min_{X'_{cj} \in \mathcal{S}_{cj}} V(X'_{cj})$. Moreover, by assumption P_c can steer the system state X_{cj} to an ERP winning state $X_{cj}^{t^\star}$ within the time period T by using the strategy U_c^1. Thus, if the current system state $X_{cj} \in \mathcal{S}_{cj}$ is such that $\varrho(X_{cj}; f_c) > \max\{-VT, 0\}$, then the positive safe distance is kept before $X_{cj}^{t^\star}$ is reached. Thus, by Definition 3, X_{cj} is an ERP winning state. If U_c^2 is an ERP winning strategy for $X_{cj}^{t^\star}$, then the strategy pair (U_c^1, U_c^2) can form an ERP winning strategy for X_{cj} as described. □

Remark 1 *By the proof of Theorem 2, the optimal value V to the problem* (7) *is the minimum speed of the region $\mathbb{E}(X'_{cj}; f_c)$ moving away from Ω_{goal} for all $X'_{cj} \in \mathcal{S}_{cj}$. Moreover, the nonlinear program* (7) *can be largely simplified if the derivatives of the PEFs are easy to compute.*

5 Positional Pursuit Enclosure Function

This section introduces a class of PEFs based on players' positions which extend the potential function in Yan et al. (2022) to the plane. We present an initial set of the induced ERP winning states and then generate more based on them using the steer-to-ERP approach. Since the coalition reduction in Theorem 1 shows that at most two pursuers are needed to ensure an ERP winning against an evader, we present the parameters, states and strategies that can ensure the ERP winning for the cases of one pursuer and two pursuers, respectively.

5.1 Positional PEFs

The following lemma identifies the positional PEF.

Lemma 2 (Positional PEF). *For $P_i \in \mathscr{P}$ and $E_j \in \mathscr{E}$, the function $f_i^{\text{ps}} : \mathbb{R}^2 \times \mathcal{S}_{ij} \to \mathbb{R}$ defined by*

$$f_i^{\text{ps}}(\boldsymbol{x}, X_{ij}) = \|\boldsymbol{x} - \boldsymbol{x}_{P_i}\|_2 - \alpha_{ij}\|\boldsymbol{x} - \boldsymbol{x}_{E_j}\|_2 - r_i \tag{12}$$

is a PEF. We call f_i^{ps} the positional PEF.

Proof. We prove that f_i^{ps} is a PEF by verifying the conditions (1)-(3) in Definition 1. Regarding the condition (2), if $\boldsymbol{x} = \boldsymbol{x}_{P_i}$, then $f_i^{\text{ps}} = -\alpha_{ij}\|\boldsymbol{x}_{P_i} - \boldsymbol{x}_{E_j}\|_2 - r_i < 0$. If $\boldsymbol{x} = \boldsymbol{x}_{E_j}$, then $f_i^{\text{ps}} = \|\boldsymbol{x}_{E_j} - \boldsymbol{x}_{P_i}\|_2 - r_i > 0$ noting $X_{ij} \in \mathcal{S}_{ij}$. Therefore, if $f_i^{\text{ps}} = 0$, then $\boldsymbol{x} \neq \boldsymbol{x}_{P_i}$ and $\boldsymbol{x} \neq \boldsymbol{x}_{E_j}$, and thus f_i^{ps} is differentiable in $\boldsymbol{x}$, $\boldsymbol{x}_{P_i}$, θ_{P_i} and $\boldsymbol{x}_{E_j}$. Regarding the condition (3), we have $f_i^{\text{ps}}(\boldsymbol{x}_{E_j}, X_{ij}) = \|\boldsymbol{x}_{E_j} - \boldsymbol{x}_{P_i}\|_2 - r_i > 0$.

Regarding the condition (1), we build a polar coordinate system with $\boldsymbol{x}_{E_j}$ as the origin, and let $\boldsymbol{x} = \boldsymbol{x}_{E_j} + \rho\boldsymbol{e}$, where $\rho \in \mathbb{R}_{>0}$ and $\boldsymbol{e} \in \partial\mathbb{S}^2$. We parameterize $\boldsymbol{e}$ by $\boldsymbol{e} = (\cos(\psi + \psi_0), \sin(\psi + \psi_0))$, where $\psi \in [0, 2\pi)$ is the rotation with respect to positive x-axis, and $\psi_0 \in [0, 2\pi)$ is the initial rotation. Then, the boundary of $\mathbb{E}(X_{ij}; f_i^{\text{ps}})$, i.e., $f_i^{\text{ps}}(\boldsymbol{x}, X_{ij}) = 0$, in this polar coordinate becomes $\|\boldsymbol{x}_{E_j} + \rho\boldsymbol{e} - \boldsymbol{x}_{P_i}\|_2 - \alpha_{ij}\rho - r_i = 0$. Thus we have $\rho =$

$\frac{1}{\alpha_{ij}^2-1}(h_1(\psi)+h_2(\psi))$, where $h_1(\psi)$ and $h_2(\psi)$ are

$$\begin{aligned} h_1(\psi) &= (\boldsymbol{x}_{E_j}-\boldsymbol{x}_{P_i})^\top \boldsymbol{e} - \alpha_{ij} r_i \\ h_2(\psi) &= \sqrt{h_1^2(\psi) + (\alpha_{ij}^2-1)(\|\boldsymbol{x}_{E_j}-\boldsymbol{x}_{P_i}\|_2^2 - r_i^2)}. \end{aligned} \tag{13}$$

In deriving (13), $\alpha_{ij}>1$ and $\|\boldsymbol{x}_{E_j}-\boldsymbol{x}_{P_i}\|_2 > r_i$ are used, which also implies that $h_2>0$ and h_2 is real as opposed to complex. Thus, given X_{ij}, ρ is bounded and $\rho>0$, and thus $\mathbb{E}(X_{ij}; f_i^{\rm ps})$ is bounded. Since the boundary is contained in $\mathbb{E}(X_{ij}; f_i^{\rm ps})$, then it is compact. As for the strict convexity, following the same argument in the proof of (Yan et al., 2022, Lemma 3.1), we have $\rho^2 + 2(\frac{\mathrm{d}\rho}{\mathrm{d}\psi})^2 - \rho\frac{\mathrm{d}^2\rho}{\mathrm{d}\psi^2} > 0$ for all ψ. By (Yan et al., 2022, Lemma 2.1), $\mathbb{E}(X_{ij}; f_i^{\rm ps})$ is strictly convex. □

Given $X_{ij}\in\mathcal{S}_{ij}$, the locus of $f_i^{\rm ps}(\boldsymbol{x}, X_{ij})=0$ is a Cartesian oval, also called Apollonius oval (Wasz et al., 2019). Suppose that each pursuer in P_c adopts the positional PEF and let $f_c^{\rm ps} = \{f_i^{\rm ps}\}_{i\in c}$. We consider a set of states:

$$\mathcal{X}_{cj}^{\rm ps} = \{X_{cj}\in\mathcal{S}_{cj} | \varrho(X_{cj}; f_c^{\rm ps})>0, \theta_{P_i}=\sigma_i(X_{cj}), i\in c\} \tag{14}$$

where $\sigma_i:\mathcal{S}_{cj}\to[0,2\pi)$ is a heading function such that for each $X_{cj}\in\mathcal{S}_{cj}$, $[\cos\sigma_i(X_{cj}), \sin\sigma_i(X_{cj})]^\top = \frac{\boldsymbol{x}_I-\boldsymbol{x}_{P_i}}{\|\boldsymbol{x}_I-\boldsymbol{x}_{P_i}\|_2}$, where $(\boldsymbol{x}_I, \boldsymbol{x}_G)$ is the optimal solution of the convex problem $\mathcal{P}^{\rm ps}(X_{cj})$:

$$\begin{aligned} &\underset{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^2\times\mathbb{R}^2}{\text{minimize}} && d(\boldsymbol{x},\boldsymbol{y}) \\ &\text{subject to} && f_i^{\rm ps}(\boldsymbol{x}, X_{ij})\ge 0,\ g(\boldsymbol{y})\le 0,\ i\in c. \end{aligned} \tag{15}$$

Next, we present the conditions on the parameters such that $\mathcal{X}_{cj}^{\rm ps}$ is a set of ERP winning states and generate more from them via the steer-to-ERP approach, for both one-pursuer and two-pursuer cases.

5.2 ERP winning conditions for one pursuer

We introduce several notations first. We let $d_{opt_1 opt_2} = \|\boldsymbol{x}_{opt_1}-\boldsymbol{x}_{opt_2}\|_2$ (interchangeable with $d(\boldsymbol{x}_{opt_1},\boldsymbol{x}_{opt_2})$) and $\boldsymbol{e}_{opt_1 opt_2} = (\boldsymbol{x}_{opt_1}-\boldsymbol{x}_{opt_2})/d_{opt_1 opt_2}$ (if $d_{opt_1 opt_2}>0$) for $opt_1, opt_2 \in \{I, G, P_i\in\mathscr{P}, E_j\in\mathscr{E}\}$. For the goal region $\Omega_{\rm goal}$, let $\boldsymbol{e}_{IG} = g_{\boldsymbol{y}}(\boldsymbol{x}_G)/\|g_{\boldsymbol{y}}(\boldsymbol{x}_G)\|_2$ and $H(\boldsymbol{x}_G)$ be the unit gradient and the Hessian matrix of g at $\boldsymbol{x}_G$, respectively.

We first consider the case of one pursuer and one evader. Based on $\mathcal{X}_{cj}^{\rm ps}$, we present the conditions on parameters and states that can ensure the ERP winning via the steer-to-ERP approach, and give the corresponding ERP winning strategies.

Theorem 3 (ERP winning parameters, state and strategy). *Consider a one-pursuer pursuit coalition $P_c=\{P_i\}$ against an evader E_j. If the following conditions hold:*

(1) the parameters satisfy

$$\alpha_{ij}>3, \quad r_i/\kappa_i > \mathrm{CM}_1(\alpha_{ij}) \tag{16}$$

where $\mathrm{CM}_1(\alpha_{ij})$ is a bound of the ratio between the capture radius and the minimum turning radius:

$$\mathrm{CM}_1(\alpha_{ij}) = 1 + \frac{3\alpha_{ij}^2+4\alpha_{ij}-3}{(\alpha_{ij}-1)^2(\alpha_{ij}-3)}; \tag{17}$$

(2) the safe distance of the state $X_{ij}\in\mathcal{S}_{ij}$ satisfies

$$\varrho(X_{ij}; f_i^{\rm ps}) > \frac{2\pi r_i/(\alpha_{ij}-1)}{r_i/\kappa_i - \mathrm{CM}_1(\alpha_{ij})} \tag{18}$$

then X_{ij} is an ERP winning state and the feedback pursuit strategy

$$u_{P_i} = \begin{cases} -\frac{\kappa_i}{v_{P_i}}\frac{\boldsymbol{e}_{IP_i}^{\circ\top}\dot{\boldsymbol{x}}_I}{d_{IP_i}}, & \text{if } \theta_{P_i}=\sigma_i(X_{ij}) \\ \mathrm{sgn}(\sin(\sigma_i(X_{ij})-\theta_{P_i})), & \text{otherwise} \end{cases} \tag{19}$$

for $X_{ij}\in\mathcal{S}_{ij}$, is an ERP winning strategy with $(\boldsymbol{x}_I,\boldsymbol{x}_G)$ computed by the convex optimization problem (15) and

$$\dot{\boldsymbol{x}}_I = \frac{(\alpha_{ij}\boldsymbol{a}_1^\circ\boldsymbol{e}_{IE_j}^\top + A)\dot{\boldsymbol{x}}_{E_j} - v_{P_i}\boldsymbol{a}_1^\circ}{d_f\boldsymbol{a}_1^\top\boldsymbol{e}_{IG}^\circ}$$

where $d_f = \|\boldsymbol{e}_{IP_i} - \alpha_{ij}\boldsymbol{e}_{IE_j}\|_2$ and

$$\begin{aligned} \boldsymbol{a}_1 &= \frac{\boldsymbol{e}_{IP_i}^\circ\boldsymbol{e}_{IP_i}^\top\boldsymbol{e}_{IG}}{d_{IP_i}} - \frac{\alpha_{ij}\boldsymbol{e}_{IE_j}^\circ\boldsymbol{e}_{IE_j}^\top\boldsymbol{e}_{IG}}{d_{IE_j}} - \frac{d_f a_2\boldsymbol{e}_{IG}^\circ}{1+d_{IG}a_2} \\ A &= \frac{\alpha_{ij}d_f\boldsymbol{e}_{IG}^\circ\boldsymbol{e}_{IG}^\top\boldsymbol{e}_{IE_j}^\circ\boldsymbol{e}_{IE_j}^\top}{d_{IE_j}},\ a_2 = \frac{\boldsymbol{e}_{IG}^{\circ\top}H(\boldsymbol{x}_G)\boldsymbol{e}_{IG}^\circ}{\|g_{\boldsymbol{y}}(\boldsymbol{x}_G)\|_2}. \end{aligned} \tag{20}$$

Proof. We prove the theorem by first proving that $\mathcal{X}_{ij}^{\rm ps}$ in (14) is a set of ERP winning states, where the key is to show that (14) is closed under the strategy (19). Additionally, we spend much space to obtain the parameter condition (16) to ensure that the strategy (19) is feasible, i.e., $|u_{P_i}|\le 1$. Finally, we prove that the states meeting (18) can be generated by the ERP winning states in $\mathcal{X}_{ij}^{\rm ps}$, via the steer-to-ERP approach in Theorem 2, provided that (16) holds.

For simplicity, the subscripts i and j will be omitted in the proof. We first prove that $\mathcal{X}^{\rm ps}$ in (14) is a set of ERP winning states.

We first prove that $\theta_P\equiv\sigma(X)$ under the strategy (19) if it holds initially. By the definition of σ above (15) and the dynamics (1), this equivalently implies that

$$\dot{\boldsymbol{x}}_P \equiv v_P\boldsymbol{e}_{IP}, \tag{21}$$

i.e., P's heading is always pointing at the point $\boldsymbol{x}_I$. Thus, we only need to verify (21) under the strategy (19) given it holds initially. Taking the time derivative for (21), we have

$$\begin{aligned}
&\frac{\mathrm{d}\dot{\boldsymbol{x}}_P}{\mathrm{d}t} - v_P\frac{\mathrm{d}\boldsymbol{e}_{IP}}{\mathrm{d}t} \\
&= -\dot{\boldsymbol{x}}_P^{\circ}\dot{\theta}_P - v_P\frac{\dot{\boldsymbol{x}}_I - \dot{\boldsymbol{x}}_P - \boldsymbol{e}_{IP}\boldsymbol{e}_{IP}^{\top}(\dot{\boldsymbol{x}}_I - \dot{\boldsymbol{x}}_P)}{d_{IP}} \\
&= v_P\boldsymbol{e}_{IP}^{\circ}\frac{v_P}{\kappa}\frac{\kappa}{v_P}\frac{\boldsymbol{e}_{IP}^{\circ\top}\dot{\boldsymbol{x}}_I}{d_{IP}} - v_P\frac{\dot{\boldsymbol{x}}_I - \boldsymbol{e}_{IP}\boldsymbol{e}_{IP}^{\top}\dot{\boldsymbol{x}}_I}{d_{IP}} = 0,
\end{aligned} \tag{22}$$

where the first equality follows from the definitions of $\dot{\boldsymbol{x}}_P$ and $\boldsymbol{e}_{IP}$, and the second equality follows from (21), the dynamics (1) and the strategy (19) for $\theta_P = \sigma(X)$. Next, we show that if $X \in \mathcal{X}^{\mathrm{ps}}$, then $\varrho(X^t; f^{\mathrm{ps}}) > 0$ for all $t \geq 0$. It suffices to prove that the speed of $\mathbb{E}(X; f^{\mathrm{ps}})$ moving away from Ω_{goal} is non-negative, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X; f^{\mathrm{ps}}) \geq 0$, for all $X \in \mathcal{X}^{\mathrm{ps}}$. According to (10) and (11), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X; f^{\mathrm{ps}}) = \frac{\mathrm{d}}{\mathrm{d}t}d(\boldsymbol{x}_I, \boldsymbol{x}_G) \\
&= \lambda(f_P^{\mathrm{ps}\top}(\boldsymbol{x}_I, X)\dot{\boldsymbol{x}}_P + v_E f_E^{\mathrm{ps}\top}(\boldsymbol{x}_I, X)\boldsymbol{u}_E) \\
&= \lambda(-v_P + v_E\alpha\boldsymbol{e}_{IE}^{\top}\boldsymbol{u}_E) \geq \lambda(-v_P + v_E\alpha) = 0,
\end{aligned}$$

where $\lambda \leq 0$ is the Lagrange multiplier, $f_P^{\mathrm{ps}} = \partial f^{\mathrm{ps}}/\partial\boldsymbol{x}_P$ and $f_E^{\mathrm{ps}} = \partial f^{\mathrm{ps}}/\partial\boldsymbol{x}_E$, and the third equality follows from (12) and (21). From the above, $X^t \in \mathcal{X}^{\mathrm{ps}}$ for all $t \geq 0$ and all $X \in \mathcal{X}^{\mathrm{ps}}$, under the strategy (19).

In order to implement the strategy (19) for $\theta_P = \sigma(X)$, we need to ensure $|u_P| \leq 1$. To that end, we present the computation of $\dot{\boldsymbol{x}}_I$. Note that $(\boldsymbol{x}_I, \boldsymbol{x}_G)$ satisfies the KKT conditions (8) consistently. For (8b), we have

$$\begin{aligned}
&g_{\boldsymbol{y}}(\boldsymbol{x}_G) \,||\, (\boldsymbol{x}_I - \boldsymbol{x}_G) \Rightarrow g_{\boldsymbol{y}}^{\top}(\boldsymbol{x}_G)(\boldsymbol{x}_I - \boldsymbol{x}_G)^{\circ} \equiv 0 \\
&\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(g_{\boldsymbol{y}}^{\top}(\boldsymbol{x}_G)(\boldsymbol{x}_I - \boldsymbol{x}_G)^{\circ}\right) = 0 \Rightarrow \\
&g_{\boldsymbol{y}}^{\circ\top}(\boldsymbol{x}_G)\dot{\boldsymbol{x}}_I = (H^{\top}(\boldsymbol{x}_G)(\boldsymbol{x}_I - \boldsymbol{x}_G)^{\circ} + g_{\boldsymbol{y}}^{\circ}(\boldsymbol{x}_G))^{\top}\dot{\boldsymbol{x}}_G,
\end{aligned} \tag{23}$$

where $||$ denotes the parallel of two vectors. For (8d), we have

$$g(\boldsymbol{x}_G) \equiv 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}g(\boldsymbol{x}_G) = 0 \Rightarrow g_{\boldsymbol{y}}^{\top}(\boldsymbol{x}_G)\dot{\boldsymbol{x}}_G = 0. \tag{24}$$

By (23) and (24), $\dot{\boldsymbol{x}}_G$ is computed by

$$\begin{aligned}
\dot{\boldsymbol{x}}_G &= \frac{g_{\boldsymbol{y}}^{\circ}(\boldsymbol{x}_G)g_{\boldsymbol{y}}^{\circ\top}(\boldsymbol{x}_G)\dot{\boldsymbol{x}}_I}{\|g_{\boldsymbol{y}}(\boldsymbol{x}_G)\|_2^2 + (\boldsymbol{x}_I - \boldsymbol{x}_G)^{\circ\top}H(\boldsymbol{x}_G)g_{\boldsymbol{y}}^{\circ}(\boldsymbol{x}_G)} \\
&= \frac{\|g_{\boldsymbol{y}}(\boldsymbol{x}_G)\|_2^2\boldsymbol{e}_{IG}^{\circ}\boldsymbol{e}_{IG}^{\circ\top}\dot{\boldsymbol{x}}_I}{\|g_{\boldsymbol{y}}(\boldsymbol{x}_G)\|_2^2 + d_{IG}\|g_{\boldsymbol{y}}(\boldsymbol{x}_G)\|_2\boldsymbol{e}_{IG}^{\circ\top}H(\boldsymbol{x}_G)\boldsymbol{e}_{IG}^{\circ}} \\
&= \frac{\boldsymbol{e}_{IG}^{\circ}\boldsymbol{e}_{IG}^{\circ\top}\dot{\boldsymbol{x}}_I}{1 + d_{IG}a_2},
\end{aligned} \tag{25}$$

where a_2 is defined in (20), and the second equality is due to the fact that $g_{\boldsymbol{y}}^{\circ}(\boldsymbol{x}_G) = \|g_{\boldsymbol{y}}(\boldsymbol{x}_G)\|_2\boldsymbol{e}_{IG}^{\circ}$ using (23) and (8d). By combining (8a) and (8b), we can obtain that $f_{\boldsymbol{x}}^{\mathrm{ps}}(\boldsymbol{x}_I, X) \,||\, g_{\boldsymbol{y}}(\boldsymbol{x}_G)$, i.e., $f_{\boldsymbol{x}}^{\mathrm{ps}\top}(\boldsymbol{x}_I, X)g_{\boldsymbol{y}}^{\circ}(\boldsymbol{x}_G) \equiv 0$, where $f_{\boldsymbol{x}}^{\mathrm{ps}} = \partial f^{\mathrm{ps}}/\partial\boldsymbol{x}$. Thus $\frac{\mathrm{d}}{\mathrm{d}t}(f_{\boldsymbol{x}}^{\mathrm{ps}\top}(\boldsymbol{x}_I, X)g_{\boldsymbol{y}}^{\circ}(\boldsymbol{x}_G)) = 0$ and thus

$$\boldsymbol{c}_1^{\top}\dot{\boldsymbol{x}}_I + \boldsymbol{c}_2^{\top}\dot{\boldsymbol{x}}_P + \boldsymbol{c}_3^{\top}\dot{\boldsymbol{x}}_E + \boldsymbol{c}_4^{\top}\dot{\boldsymbol{x}}_G = 0, \tag{26}$$

where

$$\begin{aligned}
&\boldsymbol{c}_1 = \frac{\boldsymbol{e}_{IP}^{\circ}\boldsymbol{e}_{IP}^{\top}\boldsymbol{e}_{IG}}{d_{IP}} - \frac{\alpha\boldsymbol{e}_{IE}^{\circ}\boldsymbol{e}_{IE}^{\top}\boldsymbol{e}_{IG}}{d_{IE}},\ \boldsymbol{c}_2 = -\frac{\boldsymbol{e}_{IP}^{\circ}\boldsymbol{e}_{IP}^{\top}\boldsymbol{e}_{IG}}{d_{IP}} \\
&\boldsymbol{c}_3 = \frac{\alpha\boldsymbol{e}_{IE}^{\circ}\boldsymbol{e}_{IE}^{\top}\boldsymbol{e}_{IG}}{d_{IE}},\ \boldsymbol{c}_4 = -\frac{d_f H^{\top}(\boldsymbol{x}_G)\boldsymbol{e}_{IG}^{\circ}}{\|g_{\boldsymbol{y}}(\boldsymbol{x}_G)\|_2}.
\end{aligned}$$

In order to obtain (26), (12) and $g_{\boldsymbol{y}}^{\circ}(\boldsymbol{x}_G) = \|g_{\boldsymbol{y}}(\boldsymbol{x}_G)\|_2\boldsymbol{e}_{IG}^{\circ}$ are both used. For (8c), we have $f^{\mathrm{ps}}(\boldsymbol{x}_I, X) \equiv 0$, that is, $\frac{\mathrm{d}}{\mathrm{d}t}f^{\mathrm{ps}}(\boldsymbol{x}_I, X) = 0$, leading to

$$d_f\boldsymbol{e}_{IG}^{\top}\dot{\boldsymbol{x}}_I - \boldsymbol{e}_{IP}^{\top}\dot{\boldsymbol{x}}_P + \alpha\boldsymbol{e}_{IE}^{\top}\dot{\boldsymbol{x}}_E = 0, \tag{27}$$

where (8a) is used, i.e., $f_{\boldsymbol{x}}^{\mathrm{ps}}(\boldsymbol{x}_I, X) = d_f\boldsymbol{e}_{IG}$, and $d_f = \|\boldsymbol{e}_{IP} - \alpha\boldsymbol{e}_{IE}\|_2$. By substituting (25) into (26) and combining (27), $\dot{\boldsymbol{x}}_I$ satisfies

$$\begin{bmatrix} \boldsymbol{c}_1^{\top} + \boldsymbol{c}_5^{\top} \\ d_f\boldsymbol{e}_{IG}^{\top} \end{bmatrix}\dot{\boldsymbol{x}}_I + \begin{bmatrix} \boldsymbol{c}_2^{\top}\dot{\boldsymbol{x}}_P + \boldsymbol{c}_3^{\top}\dot{\boldsymbol{x}}_E \\ \alpha\boldsymbol{e}_{IE}^{\top}\dot{\boldsymbol{x}}_E - \boldsymbol{e}_{IP}^{\top}\dot{\boldsymbol{x}}_P \end{bmatrix} = \mathbf{0}, \tag{28}$$

from which

$$\dot{\boldsymbol{x}}_I = \frac{k_1(\boldsymbol{c}_1 + \boldsymbol{c}_5)^{\circ} - k_2 d_f\boldsymbol{e}_{IG}^{\circ}}{k_3}, \tag{29}$$

where

$$\begin{aligned}
&k_1 = \alpha\boldsymbol{e}_{IE}^{\top}\dot{\boldsymbol{x}}_E - \boldsymbol{e}_{IP}^{\top}\dot{\boldsymbol{x}}_P, \quad k_2 = \boldsymbol{c}_2^{\top}\dot{\boldsymbol{x}}_P + \boldsymbol{c}_3^{\top}\dot{\boldsymbol{x}}_E \\
&k_3 = d_f(\boldsymbol{c}_1 + \boldsymbol{c}_5)^{\top}\boldsymbol{e}_{IG}^{\circ}, \quad \boldsymbol{c}_5 = -\frac{d_f a_2\boldsymbol{e}_{IG}^{\circ}}{1 + d_{IG}a_2},
\end{aligned}$$

where the definition of a_2 in (20) is used. Note that

$$|k_1| \leq 2v_P, \quad |k_2| \leq v_P/d_{IP} + v_P/d_{IE}. \tag{30}$$

Let $\gamma_{PG} = \boldsymbol{e}_{IP}^{\top}\boldsymbol{e}_{IG}$ and $\gamma_{EG} = \boldsymbol{e}_{IE}^{\top}\boldsymbol{e}_{IG}$, then k_3 can be rewritten as follows

$$k_3 = \frac{d_f}{d_{IP}}\left(\gamma_{PG}^2 - \frac{\alpha d_{IP}\gamma_{EG}^2}{d_{IE}}\right) - d_f\|\boldsymbol{c}_5\|_2. \tag{31}$$

Since $f_{\boldsymbol{x}}^{\mathrm{ps}}(\boldsymbol{x}_I, X) = d_f\boldsymbol{e}_{IG}$, this implies that

$$d_f = (\boldsymbol{e}_{IP} - \alpha\boldsymbol{e}_{IE})^{\top}\boldsymbol{e}_{IG} = \gamma_{PG} - \alpha\gamma_{EG}. \tag{32}$$

Since $d_f \in [\alpha-1, \alpha+1]$ by definition and $f^{\text{ps}}(\boldsymbol{x}_I, X) = 0$ leads to $d_{IP} = \alpha d_{IE} + r$, by combining them with (31) and (32) we have the inequality

$$\begin{aligned}
&\gamma_{PG}^2 - \frac{\alpha d_{IP}\gamma_{EG}^2}{d_{IE}} = \gamma_{PG}^2 - \frac{(\alpha d_{IE}+r)(d_f-\gamma_{PG})^2}{\alpha d_{IE}} \\
&= \frac{-r\gamma_{PG}^2 + 2d_f(\alpha d_{IE}+r)\gamma_{PG} - (\alpha d_{IE}+r)d_f^2}{\alpha d_{IE}} \\
&\leq -\frac{r(d_f-1)^2 + \alpha d_f d_{IE}(d_f-2)}{\alpha d_{IE}} := -(rk_4/d_{IE}+k_5),
\end{aligned} \tag{33}$$

where $\alpha > 3$ by assumption in (16) is used in the last inequality which follows when $\gamma_{PG} = 1$, and

$$k_4 = (d_f-1)^2/\alpha, \quad k_5 = d_f(d_f-2). \tag{34}$$

Since $k_4, k_5 > 0$, then (33) implies that $|k_3|$ in (31) has a positive lower bound

$$|k_3| \geq (rk_4/d_{IE}+k_5)d_f/d_{IP} + d_f\|\boldsymbol{c}_5\|_2. \tag{35}$$

Using (29) and the bounds for k_1, k_2 and k_3 in (30) and (35), we can derive an upper bound for $\|\dot{\boldsymbol{x}}_I\|_2$ as follows

$$\begin{aligned}
\|\dot{\boldsymbol{x}}_I\|_2 &\leq \frac{2v_P(\|\boldsymbol{c}_1\|_2+\|\boldsymbol{c}_5\|_2) + d_f(v_P/d_{IP}+v_P/d_{IE})}{(rk_4/d_{IE}+k_5)d_f/d_{IP} + d_f\|\boldsymbol{c}_5\|_2} \\
&= \frac{2v_P d_{IP}(\|\boldsymbol{c}_1\|_2+\|\boldsymbol{c}_5\|_2) + d_f v_P + d_f d_{IP} v_P/d_{IE}}{(rk_4/d_{IE}+k_5)d_f + d_{IP}d_f\|\boldsymbol{c}_5\|_2} \\
&\leq \frac{2v_P}{d_f} + \frac{rk_6/(\alpha d_{IE}) + k_7}{(rk_4/d_{IE}+k_5)d_f + d_{IP}d_f\|\boldsymbol{c}_5\|_2} v_P,
\end{aligned} \tag{36}$$

where k_6 and k_7 are respectively given by

$$\begin{aligned}
k_6 &= -2d_f^2 + (\alpha+4)d_f + 2\alpha^2 - 2 \geq \alpha(\alpha+1) > 0 \\
k_7 &= -2d_f^2 + (\alpha+5)d_f + 2\alpha^2 + 2 \geq k_6 > 0,
\end{aligned}$$

In (36), $\|\boldsymbol{c}_1\|_2 \leq 1/d_{IP} + \alpha/d_{IE}$ and $d_{IP} = \alpha d_{IE} + r$ are used, and $k_6 \geq \alpha(\alpha+1)$ is due to $d_f \in [\alpha-1, \alpha+1]$. Then, the upper bound is further derived as follows

$$\begin{aligned}
\|\dot{\boldsymbol{x}}_I\|_2 &\leq \frac{2v_P}{d_f} + \frac{rk_6/(\alpha d_{IE}) + k_7}{(rk_4/d_{IE}+k_5)d_f} v_P \\
&\leq \frac{2v_P}{\alpha-1} + v_P \max\{k_6/(\alpha d_f k_4), k_7/(d_f k_5)\},
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
\frac{k_6}{\alpha d_f k_4} &= \frac{2\alpha^2}{d_f(d_f-1)^2} + \frac{\alpha}{(d_f-1)^2} - \frac{2}{d_f} \\
&\leq \frac{2\alpha^2}{(\alpha-1)(\alpha-2)^2} + \frac{\alpha}{(\alpha-2)^2} - \frac{2}{\alpha+1} \\
&= \frac{\alpha^3 + 12\alpha^2 - 17\alpha + 8}{(\alpha^2-1)(\alpha-2)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{k_7}{d_f k_5} &= \frac{2\alpha^2+2}{d_f^2(d_f-2)} + \frac{\alpha+5}{d_f(d_f-2)} - \frac{2}{d_f-2} \\
&\leq \frac{2\alpha^2+2}{(\alpha-1)^2(\alpha-3)} + \frac{\alpha+5}{(\alpha-1)(\alpha-3)} - \frac{2}{\alpha-1} \\
&= \frac{\alpha^2+12\alpha-9}{(\alpha-1)^2(\alpha-3)}.
\end{aligned}$$

where $d_f \in [\alpha-1, \alpha+1]$ and the definitions of k_4, k_5, k_6 and k_7 are used. Note that the upper bound of $k_7/(d_f k_5)$ is larger than the upper bound of $k_6/(\alpha d_f k_4)$. Thus, the upper bound (37) becomes

$$\|\dot{\boldsymbol{x}}_I\|_2 \leq \frac{3\alpha^2+4\alpha-3}{(\alpha-1)^2(\alpha-3)} v_P. \tag{38}$$

By combining (38) and (19), the control u_P for $\theta_P = \sigma(X)$ has the following bound

$$|u_P| = \Big| -\frac{\kappa}{v_P}\frac{\boldsymbol{e}_{IP}^{\circ\top}\dot{\boldsymbol{x}}_I}{d_{IP}} \Big| \leq \frac{\kappa\|\dot{\boldsymbol{x}}_I\|_2}{v_P r} \leq \frac{\kappa(3\alpha^2+4\alpha-3)}{r(\alpha-1)^2(\alpha-3)},$$

where $d_{IP} \geq r$ is used, which implies that $|u_P| \leq 1$ holds under the parameter condition (16). Thus, $\mathcal{X}^{\text{ps}}$ is a set of ERP winning states.

By Theorem 2, we next prove that the states meeting (18) can be generated by the ERP winning states in $\mathcal{X}^{\text{ps}}$, via the steer-to-ERP approach. To that end, we compute such a time horizon T in Theorem 2 via a lower bound of the angle chasing speed $|\dot{\theta}_P| - |\dot{\sigma}|$. Since the speed of the heading function σ is bounded by

$$|\dot{\sigma}| \leq \frac{\|\dot{\boldsymbol{x}}_I\|_2 + \|\dot{\boldsymbol{x}}_P\|_2}{d_{IP}} \leq \frac{\|\dot{\boldsymbol{x}}_I\|_2 + v_P}{r}, \tag{39}$$

then under the strategy (19) for $\theta_P \neq \sigma(X)$ we have

$$\begin{aligned}
|\dot{\theta}_P| - |\dot{\sigma}| &\geq v_P/\kappa - (\|\dot{\boldsymbol{x}}_I\|_2 + v_P)/r \\
&\geq \frac{v_P}{r}\Big(\frac{r}{\kappa} - 1 - \frac{3\alpha^2+4\alpha-3}{(\alpha-1)^2(\alpha-3)}\Big) > 0,
\end{aligned} \tag{40}$$

where (38) and (16) are used. Since the maximum value of $|\theta_P - \sigma|$ to reduce under (19) for $\theta_P \neq \sigma(X)$ is π, any initial state will meet the condition $\theta_P = \sigma(X)$ within at most time $T := \pi/(|\dot{\theta}_P| - |\dot{\sigma}|)$. In order to ensure the positive safe distance before $\theta_P = \sigma(X)$, we need to compute the speed $\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X; f^{\text{ps}})$ of $\mathbb{E}(X; f^{\text{ps}})$ moving away from Ω_{goal}. The KKT condition (8a) implies that $|\lambda| = 1/d_f$. By Theorem 2 and Remark 1, $\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X; f^{\text{ps}})$ is the optimal value of (7) and is bounded by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X; f^{\text{ps}}) &\geq -|\lambda|(v_P|f_P^{\text{ps}\top}\boldsymbol{e}_{\theta_P}| + v_E\|f_E^{\text{ps}}\|_2) \\
&\geq -(v_P + v_E\alpha)/d_f \geq 2v_P/(1-\alpha).
\end{aligned} \tag{41}$$

Therefore, using (40) and (41), if a state $X \in \mathcal{S}$ satisfies

$$\varrho(X; f^{\mathrm{ps}}) > \frac{2\pi r}{(\alpha - 1)(r/\kappa - \mathrm{CM}_1(\alpha))}, \tag{42}$$

then $\varrho(X; f^{\mathrm{ps}}) + T \min_{t' \in [0,T]} \frac{\mathrm{d}}{\mathrm{d}t} \varrho(X^{t'}; f^{\mathrm{ps}}) > 0$, implying that $\varrho(X^{t'}; f^{\mathrm{ps}}) > 0$ for all $t' \in [0, t^\star]$ and $X^{t^\star}$ is an ERP winning state in $\mathcal{X}^{\mathrm{ps}}$ for some $t^\star \in [0, T]$. □

Remark 2 *By* (19), (21) *and* (22), *in order to ensure that* $\theta_{P_i} = \sigma_i(X_{ij})$ *holds once reached, the current control input of* E_j *is required to compute* $\dot{\boldsymbol{x}}_I$ *as it involves the term* $\dot{\boldsymbol{x}}_{E_j}$. *In practice, if the methods in Section 2.2 are used to estimate or measure* $\dot{\boldsymbol{x}}_{E_j}$ *and the errors are not negligible, then we need to relax* $\theta_{P_i} = \sigma_i(X_{ij})$ *based on the error bounds to generalise* $\mathcal{X}_{ij}^{\mathrm{ps}}$ *in* (14), *for which a possibly looser winning condition and a robust strategy might be obtained. Since the related robustness analysis is not straightforward, we leave it for future work.*

The proof of Theorem 3 shows that $\mathcal{X}_{ij}^{\mathrm{ps}}$ is a set of ERP winning states under the parameters (16) which can be relaxed further as below.

Lemma 3 (Relaxed ERP winning parameters). *If the state satisfies* $X_{ij} \in \mathcal{X}_{ij}^{\mathrm{ps}}$ *and the parameters satisfy*

$$\alpha_{ij} > 3, \; r_i/\kappa_i \geq \frac{(5\alpha_{ij}^2 - 9\alpha_{ij} + 8)}{(\alpha_{ij} - 1)(\alpha_{ij} - 2)^2} \tag{43}$$

then X_{ij} *is an ERP winning state and the strategy* (19) *is an ERP winning strategy.*

Proof. For simplicity, the subscripts i and j will be omitted in the proof. Since $X \in \mathcal{X}^{\mathrm{ps}}$, then (21) holds and thus the control u_P for $\theta_P = \sigma(X)$ in (19) satisfies

$$\begin{aligned}
|u_P| &= \Big| \frac{\kappa}{v_P} \frac{\boldsymbol{e}_{IP}^{\circ\top} \dot{\boldsymbol{x}}_I}{d_{IP}} \Big| = \frac{\kappa |k_1 \boldsymbol{e}_{IP}^\top (\boldsymbol{c}_5 - \boldsymbol{c}_3) - k_2 d_f \boldsymbol{e}_{IP}^\top \boldsymbol{e}_{IG}|}{v_P d_{IP} |k_3|} \\
&\leq \frac{\kappa (2 v_P (\|\boldsymbol{c}_5\|_2 + \alpha/d_{IE}) + v_P d_f / d_{IE})}{v_P ((r k_4 / d_{IE} + k_5) d_f + d_f d_{IP} \|\boldsymbol{c}_5\|_2)} \\
&= \frac{2\kappa}{d_f d_{IP}} + \frac{\kappa((2\alpha + d_f)/d_{IE} - 2(r k_4/d_{IE} + k_5)/d_{IP})}{(r k_4/d_{IE} + k_5) d_f + d_f d_{IP} \|\boldsymbol{c}_5\|_2} \\
&\leq \frac{2\kappa}{d_f d_{IP}} + \frac{\kappa(2\alpha + d_f)/d_{IE}}{(r k_4/d_{IE} + k_5) d_f} \leq \frac{2\kappa}{d_f r} + \frac{\kappa(2\alpha + d_f)}{r k_4 d_f} \\
&= \frac{\kappa}{r} \Big(\frac{2}{d_f} + \frac{\alpha(2\alpha + d_f)}{d_f (d_f - 1)^2} \Big) \leq \frac{\kappa(5\alpha^2 - 9\alpha + 8)}{r(\alpha - 1)(\alpha - 2)^2} \leq 1,
\end{aligned} \tag{44}$$

where the second equality is due to (29), the first inequality follows from (30), (35) and $\boldsymbol{e}_{IP}^{\circ\top} \dot{\boldsymbol{x}}_P = 0$ by (21), the third inequality is based on the fact that $d_{IP} \geq r$ and $d_{IE} > 0$, the fourth inequality is because the formulation is monotonically decreasing in d_f which has the range $[\alpha - 1, \alpha + 1]$, and the last inequality holds using the condition (43). Therefore, the conclusion follows by the argument in the proof of Theorem 3. □

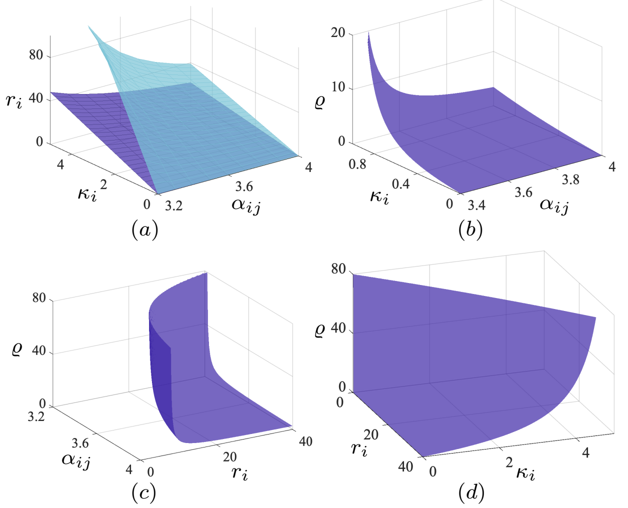

Fig. 3. The ERP winning conditions for one pursuer and one evader with the PEF (12). (a) parameters in (16) (purple) and (43) (blue); safe distance in (18): (b) $r_i = 20$, (c) $\kappa_i = 1$, (d) $\alpha_{ij} = 4$.

The ERP winning conditions in Theorem 3 and Lemma 3 for one pursuer and one evader with the PEF (12) are shown in Fig. 3. In Fig. 3(a), the boundaries of winning parameters (i.e., α_{ij}, r_i and κ_i) given by (16) and (43) are depicted in purple and blue, respectively. The boundary of the winning condition (18) on the safe distance and parameters is depicted in Fig. 3(b)-(d) when the value of one of the parameters is fixed.

5.3 ERP winning conditions for two pursuers

We next consider two pursuers and one evader. Based on $\mathcal{X}_{cj}^{\mathrm{ps}}$ and results of the one-pursuer case, we present the conditions on parameters and states that can ensure the ERP winning via the steer-to-ERP approach, and give the corresponding ERP winning strategies.

Theorem 4 (ERP winning parameters, state and strategy). *Let* $c = \{1, 2\}$. *Consider a two-pursuer pursuit coalition* $P_c = \{P_1, P_2\}$ *against an evader* E_j. *There are two scenarios for* $X_{cj} \in \mathcal{S}_{cj}$:

(1) *if* $\mathcal{P}^{\mathrm{ps}}(X_{cj})$ *has the unique support constraint (say* P_1*), then it goes to one-pursuer case in Theorem 3. Verify* P_1 *against* E_j *first, and if it fails, then verify* P_2 *against* E_j;

(2) *if* $\mathcal{P}^{\mathrm{ps}}(X_{cj})$ *has two support constraints, then: if the parameters* (16) *hold for* P_1 *and* P_2 *against* E_j *separately and the safe distance of* X_{cj} *satisfies*

$$\varrho(X_{cj}; f_c^{\mathrm{ps}}) > \min_{i' \in c} \max_{i \in c} \frac{2\pi_i r_i v_{P_{i'}} / (\alpha_{i'} v_{P_i} - v_{P_i})}{r_i/\kappa_i - \mathrm{CM}_2(\alpha_1, \alpha_2)}, \tag{45}$$

where

$$\mathrm{CM}_2(\alpha_1,\alpha_2) = 1 + \min_{i''\in c} \frac{3\alpha_{i''}^2 + 4\alpha_{i''} - 3}{(\alpha_{i''}-1)^2(\alpha_{i''}-3)},$$

then X_{cj} is an ERP winning state and the cooperative feedback pursuit strategy

$$u_{P_i} = \begin{cases} -\frac{\kappa_i}{v_{P_i}}\frac{e_{IP_i}^{\circ\top}\dot{x}_I}{d_{IP_i}}, \text{ if } \theta_{P_i} = \sigma_i(X_{cj}) \\ \mathrm{sgn}(\sin(\sigma_i(X_{cj}) - \theta_{P_i})), \text{ otherwise} \end{cases} \tag{46}$$

for all $i \in c$ and all $X_{cj} \in \mathcal{S}_{cj}$, is an ERP winning strategy with $(\boldsymbol{x}_I, \boldsymbol{x}_G)$ computed by the convex problem (15) and $\dot{\boldsymbol{x}}_I$ is given by

$$\frac{\boldsymbol{a}_1^\circ(\alpha_2 \boldsymbol{e}_{IE_j}^\top \dot{\boldsymbol{x}}_{E_j} - v_{P_2}) - \boldsymbol{a}_2^\circ(\alpha_1 \boldsymbol{e}_{IE_j}^\top \dot{\boldsymbol{x}}_{E_j} - v_{P_1})}{\boldsymbol{a}_1^\top \boldsymbol{a}_2^\circ}$$

where $\boldsymbol{a}_i = \boldsymbol{e}_{IP_i} - \alpha_{ij}\boldsymbol{e}_{IE_j}$ for $i \in c$.

Proof. For the second scenario, similar to the proof of Theorem 3, we first prove that $\mathcal{X}_{cj}^{\mathrm{ps}}$ in (14) is a set of ERP winning states by showing that (14) is closed under the strategy (46). Then we also show that the strategy (46) is feasible, i.e., $|u_{P_i}| \le 1$ for all $i \in c$ under the parameter conditions (16). Finally, we prove that the states meeting (45) can be generated by the ERP winning states in $\mathcal{X}_{cj}^{\mathrm{ps}}$, via the steer-to-ERP approach in Theorem 2.

Unless for clarity, the subscript j for E_j, $\boldsymbol{x}_{E_j}$, $\boldsymbol{u}_{E_j}$, v_{E_j}, α_{ij} and $\boldsymbol{e}_{IE_j}$ is omitted in the proof. For the first scenario, since $\mathcal{P}^{\mathrm{ps}}(X_{cj})$ has the unique support constraint (say P_1), checking the case of P_1 against E is sufficient. However, P_2 may be able to win against E even if P_1 fails. Regarding the second scenario, we first prove that $\mathcal{X}_{cj}^{\mathrm{ps}}$ is a set of ERP winning states.

We first prove that $\theta_{P_i} \equiv \sigma_i(X_{cj})$ under the strategy (46) if it holds initially. By the definition of σ_i above (15) and the dynamics (1), this equivalently implies that

$$\dot{\boldsymbol{x}}_{P_1} \equiv v_{P_1}\boldsymbol{e}_{IP_1}, \quad \dot{\boldsymbol{x}}_{P_2} \equiv v_{P_2}\boldsymbol{e}_{IP_2}. \tag{47}$$

It can be proved by following the same argument in (22). Next, we show that if $X_{cj} \in \mathcal{X}_{cj}^{\mathrm{ps}}$, then $\varrho(X_{cj}^t; f_c^{\mathrm{ps}}) > 0$ for all $t \ge 0$ under the strategy (46). It suffices to prove that the speed of $\mathbb{E}(X_{cj}; f_c^{\mathrm{ps}})$ moving away from Ω_{goal} is non-negative, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X_{cj}; f_c^{\mathrm{ps}}) \ge 0$, for all $X_{cj} \in \mathcal{X}_{cj}^{\mathrm{ps}}$. According to (10) and (11), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X_{cj}; f_c^{\mathrm{ps}}) = \frac{\mathrm{d}}{\mathrm{d}t} d(\boldsymbol{x}_I, \boldsymbol{x}_G) \\ &= \sum_{i\in c} \lambda_i (f_{i,P}^{\mathrm{ps}\top}(\boldsymbol{x}_I, X_{ij})\dot{\boldsymbol{x}}_{P_i} + v_E f_{i,E}^{\mathrm{ps}\top}(\boldsymbol{x}_I, X_{ij})\boldsymbol{u}_E) \\ &= \sum_{i\in c} \lambda_i (v_E \alpha_i \boldsymbol{e}_{IE}^\top \boldsymbol{u}_E - v_{P_i}) \ge \sum_{i\in c} \lambda_i (v_E\alpha_i - v_{P_i}) = 0. \end{aligned}$$

From the above, $X_{cj}^t \in \mathcal{X}_{cj}^{\mathrm{ps}}$ for all $t \ge 0$ and all $X_{cj} \in \mathcal{X}_{cj}^{\mathrm{ps}}$, under the strategy (46).

Similar to the one-pursuer case, in order to implement the strategy (46) for $\theta_{P_i} = \sigma_i(X_{cj})$, we also need to ensure $|u_{P_i}| \le 1$ for all $i \in c$. To that end, we present the computation of $\dot{\boldsymbol{x}}_I$ for two pursuers against one evader. Note that $(\boldsymbol{x}_I, \boldsymbol{x}_G)$ satisfies the KKT conditions (8) consistently. Since $\mathcal{P}^{\mathrm{ps}}(X_{cj})$ has two support constraints, then $\lambda_1 < 0$ and $\lambda_2 < 0$ ($c = \{1, 2\}$). Then, for (8c), we have

$$f_i^{\mathrm{ps}}(\boldsymbol{x}_I, X_{ij}) \equiv 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} f_i^{\mathrm{ps}}(\boldsymbol{x}_I, X_{ij}) = 0 \text{ for } i \in c,$$

that is,

$$\begin{bmatrix} \boldsymbol{a}_1^\top \\ \boldsymbol{a}_2^\top \end{bmatrix} \dot{\boldsymbol{x}}_I + \begin{bmatrix} \alpha_1 \boldsymbol{e}_{IE}^\top \dot{\boldsymbol{x}}_E - \boldsymbol{e}_{IP_1}^\top \dot{\boldsymbol{x}}_{P_1} \\ \alpha_2 \boldsymbol{e}_{IE}^\top \dot{\boldsymbol{x}}_E - \boldsymbol{e}_{IP_2}^\top \dot{\boldsymbol{x}}_{P_2} \end{bmatrix} = \boldsymbol{0},$$

where $\boldsymbol{a}_i = \boldsymbol{e}_{IP_i} - \alpha_i \boldsymbol{e}_{IE}$ for $i \in c$. By solving the above equations, $\dot{\boldsymbol{x}}_I$ is given by

$$\frac{\boldsymbol{a}_1^\circ(\alpha_2 \boldsymbol{e}_{IE}^\top \dot{\boldsymbol{x}}_E - \boldsymbol{e}_{IP_2}^\top \dot{\boldsymbol{x}}_{P_2}) - \boldsymbol{a}_2^\circ(\alpha_1 \boldsymbol{e}_{IE}^\top \dot{\boldsymbol{x}}_E - \boldsymbol{e}_{IP_1}^\top \dot{\boldsymbol{x}}_{P_1})}{\boldsymbol{a}_1^\top \boldsymbol{a}_2^\circ}. \tag{48}$$

Since we only need to ensure $|u_{P_i}| \le 1$ for $\theta_{P_i} = \sigma_i(X_{cj})$ and for all $i \in c$, i.e., (47) holds, then (48) is simplified as

$$\dot{\boldsymbol{x}}_I = \frac{\boldsymbol{a}_1^\circ(\alpha_2 \boldsymbol{e}_{IE}^\top \dot{\boldsymbol{x}}_E - v_{P_2}) - \boldsymbol{a}_2^\circ(\alpha_1 \boldsymbol{e}_{IE}^\top \dot{\boldsymbol{x}}_E - v_{P_1})}{\boldsymbol{a}_1^\top \boldsymbol{a}_2^\circ}. \tag{49}$$

Take $k_1 = \alpha_1 \boldsymbol{e}_{IE}^\top \dot{\boldsymbol{x}}_E - v_{P_1}$, $k_2 = \alpha_2 \boldsymbol{e}_{IE}^\top \dot{\boldsymbol{x}}_E - v_{P_2}$, $d_1 = \|\boldsymbol{a}_1\|_2$, $d_2 = \|\boldsymbol{a}_2\|_2$ and $\boldsymbol{a}_1^\top \boldsymbol{a}_2 = d_1 d_2 \cos\gamma$, where $\gamma \in (0, \pi)$. Then we have

$$\begin{aligned} &\|\dot{\boldsymbol{x}}_I\|_2^2 = \frac{k_2^2 d_1^2 + k_1^2 d_2^2 - 2k_1 k_2 d_1 d_2 \cos\gamma}{d_1^2 d_2^2 \sin^2\gamma} \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\gamma}\|\dot{\boldsymbol{x}}_I\|_2^2 = \frac{2(k_1 d_2\cos\gamma - k_2 d_1)(k_2 d_1 \cos\gamma - k_1 d_2)}{d_1^2 d_2^2 \sin^3\gamma} \\ &\Rightarrow \|\dot{\boldsymbol{x}}_I\|_2 \le \max\Big\{\lim_{\gamma\to 0}\|\dot{\boldsymbol{x}}_I\|_2, \lim_{\gamma\to\pi}\|\dot{\boldsymbol{x}}_I\|_2\Big\}. \end{aligned} \tag{50}$$

If $\gamma \to 0$, i.e., $\boldsymbol{a}_1$ is parallel to $\boldsymbol{a}_2$ with the same direction, then by (8a), one enclosure region is an inscribed region of the other and thus it is degenerated into the one pursuer case. Then by (38), we have

$$\lim_{\gamma\to 0}\|\dot{\boldsymbol{x}}_I\|_2 \le \min_{i\in c}\frac{3\alpha_i^2 + 4\alpha_i - 3}{(\alpha_i - 1)^2(\alpha_i - 3)} v_{P_i}.$$

If $\gamma \to \pi$, i.e., $\boldsymbol{a}_1$ is parallel to $\boldsymbol{a}_2$ with the opposite direction, then two enclosure regions are externally tangent. This implies that $\boldsymbol{x}_I \to \boldsymbol{x}_E$, and thus we have

$$\lim_{\gamma\to\pi}\|\dot{\boldsymbol{x}}_I\|_2 = \|\dot{\boldsymbol{x}}_E\|_2 = v_E.$$

Therefore, $\|\dot{\boldsymbol{x}}_I\|_2$ is bounded by

$$\|\dot{\boldsymbol{x}}_I\|_2 \le \min_{i\in c} \frac{3\alpha_i^2+4\alpha_i-3}{(\alpha_i-1)^2(\alpha_i-3)} v_{P_i}. \tag{51}$$

By combining (51) with (46), the control u_{P_i} for $\theta_{P_i} = \sigma_i(X_{cj})$ has the following bound

$$\begin{aligned} |u_{P_i}| &= \Big| -\frac{\kappa_i}{v_{P_i}} \frac{\boldsymbol{e}_{IP_i}^{\circ\top}\dot{\boldsymbol{x}}_I}{d_{IP_i}}\Big| \le \frac{\kappa_i\|\dot{\boldsymbol{x}}_I\|_2}{v_{P_i} r_i} \\ &\le \frac{\kappa_i}{v_{P_i} r_i} \min_{i'\in c} \frac{3\alpha_{i'}^2+4\alpha_{i'}-3}{(\alpha_{i'}-1)^2(\alpha_{i'}-3)} v_{P_{i'}} < 1, \end{aligned} \tag{52}$$

where the parameter conditions (16) are used. In conclusion, $\mathcal{X}_{cj}^{\rm ps}$ is a set of ERP winning states.

According to Theorem 2, we next prove that the states meeting (45) can be generated by the ERP winning states in $\mathcal{X}_{cj}^{\rm ps}$, via the steer-to-ERP approach. By following the same argument (39) and (40), the angle chasing speed has the positive lower bound

$$|\dot{\theta}_{P_i}| - |\dot{\sigma}_i| \ge \frac{v_{P_i}}{r_i}\Big(\frac{r_i}{\kappa_i} - 1 - \min_{i'\in c}\frac{v_{P_{i'}}(3\alpha_{i'}^2+4\alpha_{i'}-3)}{v_{P_i}(\alpha_{i'}-1)^2(\alpha_{i'}-3)}\Big).$$

Thus, similar to the one-pursuer case, any initial state will meet the condition $\theta_{P_i} = \sigma_i(X_{cj})$ for all $i\in c$ within at most time $T := \max_{i\in c} \pi/(|\dot{\theta}_{P_i}| - |\dot{\sigma}_i|)$. In order to ensure the positive safe distance before $\theta_{P_i} = \sigma_i(X_{cj})$ for all $i \in c$, we need to compute the speed $\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X_{cj}; f_c^{\rm ps})$ of $\mathbb{E}(X_{cj}; f_c^{\rm ps})$ moving away from $\Omega_{\rm goal}$. By Theorem 2 and Remark 1, $\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X_{cj}; f_c^{\rm ps})$ is the optimal value of (7) and is bounded by

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,\varrho(X_{cj}; f_c^{\rm ps}) \\ &\ge -\sum_{i\in c}|\lambda_i|(|f_{i,P}^\top \dot{\boldsymbol{x}}_{P_i}| + v_{P_i}|f_{i,\theta}|/\kappa_i + v_E\|f_{i,E}\|_2) \\ &\ge -(2v_{P_1}|\lambda_1| + 2v_{P_2}|\lambda_2|) = 2v_{P_1}\lambda_1 + 2v_{P_2}\lambda_2. \end{aligned} \tag{53}$$

Since λ_1 and λ_2 are subject to (8a), then we consider the optimization problem

$$\begin{aligned} \underset{(\lambda_1,\lambda_2)\in\mathbb{R}^2}{\text{minimize}} \quad & 2v_{P_1}\lambda_1 + 2v_{P_2}\lambda_2 \\ \text{subject to} \quad & \lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \boldsymbol{e}_{IG} = \boldsymbol{0}. \end{aligned} \tag{54}$$

The KKT conditions to (54) lead to

$$2v_{P_1} + \boldsymbol{z}^\top\boldsymbol{a}_1 = 0, \quad 2v_{P_2} + \boldsymbol{z}^\top\boldsymbol{a}_2 = 0 \tag{55a}$$
$$\lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \boldsymbol{e}_{IG} = \boldsymbol{0}, \tag{55b}$$

where $\boldsymbol{z}\in\mathbb{R}^2$ is the Lagrange multiplier, from which we

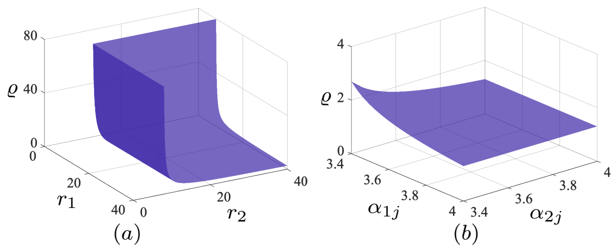

Fig. 4. The ERP winning conditions for two pursuers and one evader with the PEF (12). Safe distance in (45): (a) $\alpha_{1j} = \alpha_{2j} = 4$ and $\kappa_1 = \kappa_2 = 1$; (b) $r_1 = r_2 = 20$ and $\kappa_1 = \kappa_2 = 0.5$.

have

$$\begin{aligned} 2v_{P_1}\lambda_1 + 2v_{P_2}\lambda_2 &= -\lambda_1\boldsymbol{z}^\top\boldsymbol{a}_1 - \lambda_2\boldsymbol{z}^\top\boldsymbol{a}_2 \\ &= \boldsymbol{z}^\top\boldsymbol{e}_{IG} \ge -\|\boldsymbol{z}\|_2, \end{aligned} \tag{56}$$

and

$$\boldsymbol{z} = (2v_{P_2}\boldsymbol{a}_1^\circ - 2v_{P_1}\boldsymbol{a}_2^\circ)/(\boldsymbol{a}_1^\top\boldsymbol{a}_2^\circ). \tag{57}$$

Note that $\boldsymbol{z}$ has the similar expression as $\dot{\boldsymbol{x}}_I$ in (49). Then, following the argument (50), we obtain

$$\|\boldsymbol{z}\|_2 \le \max\Big\{\lim_{\gamma\to 0}\|\boldsymbol{z}\|_2, \lim_{\gamma\to\pi}\|\boldsymbol{z}\|_2\Big\}. \tag{58}$$

If $\gamma\to 0$, it is degenerated into the one pursuer case, and if $\gamma\to\pi$, then $\boldsymbol{x}_I \to \boldsymbol{x}_E$. This observation implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X_{cj}; f_c^{\rm ps}) \ge \max_{i\in c} 2v_{P_i}/(1-\alpha_i), \tag{59}$$

where (41) is used. Thus, if a state $X_{cj}\in\mathcal{S}_{cj}$ satisfies

$$\varrho(X_{cj}; f_c^{\rm ps}) > \min_{i'\in c}\max_{i\in c}\frac{2\pi_i r_i v_{P_{i'}}/(\alpha_{i'}v_{P_i} - v_{P_i})}{r_i/\kappa_i - \mathrm{CM}_2(\alpha_1,\alpha_2)},$$

then $\varrho(X_{cj}; f_c^{\rm ps}) + T\min_{t'\in[0,T]}\frac{\mathrm{d}}{\mathrm{d}t}\varrho(X_{cj}^{t'}; f_c^{\rm ps}) > 0$, and thus $\varrho(X_{cj}^{t'}; f_c^{\rm ps}) > 0$ for $t'\in[0,t^\star]$ and $X_{cj}^{t^\star}$ is an ERP winning state in $\mathcal{X}_{cj}^{\rm ps}$ for some $t^\star\in[0,T]$. □

The ERP winning conditions in Theorem 4 for two pursuers and one evader with the PEF (12) are shown in Fig. 4. Since the winning condition (45) on the safe distance and parameters involves seven variables, we visualise the boundaries by fixing the values of speed ratios and minimum turning radii in Fig. 4(a), and capture radii and minimum turning radii in Fig. 4(b).

6 Pursuit Strategies based on Task Allocation

This section first considers the task allocation between pursuit coalitions and evaders by piecing together the outcomes of all subgames. Let $\mathcal{G} = (\mathcal{V}_P \cup \mathcal{V}_E, \mathcal{E})$ be an undirected bipartite graph consisting of two independent vertex sets $\mathcal{V}_P$ and $\mathcal{V}_E$, and a set of edges $\mathcal{E}$. In our problem, $\mathcal{V}_P$ is the set of all nonempty pursuit coalitions of size less than or equal to two, and $\mathcal{V}_E$ the set of evaders. The edge connecting vertex $P_c \in \mathcal{V}_P$ and vertex $E_j \in \mathcal{V}_E$ is denoted by e_{cj}, and $e_{cj} \in \mathcal{E}$ if and only if P_c is able to defend against E_j through the ERP winning. Let $\mathcal{C} = (\mathcal{E}, \bar{\mathcal{E}})$ be a conflict graph, where each vertex in $\mathcal{C}$ corresponds uniquely to an edge in $\mathcal{G}$, and an edge $\bar{e} \in \bar{\mathcal{E}}$ if and only if two vertexes (two edges in $\mathcal{G}$) connecting by $\bar{e}$ involve at least one common pursuer.

We formulate the problem of maximizing the number of captured evaders in the ERP winning as a binary integer program (BIP):

$$\begin{aligned} \underset{a_{cj}, a_{pq} \in \{0,1\}}{\text{maximize}} \quad & \sum\nolimits_{e_{cj} \in \mathcal{E}} a_{cj} + z(a_{cj}) \\ \text{subject to} \quad & \sum\nolimits_{P_c \in \mathcal{V}_P} a_{cj} \leq 1, \quad \forall E_j \in \mathcal{V}_E \\ & \sum\nolimits_{E_j \in \mathcal{V}_E} a_{cj} \leq 1, \quad \forall P_c \in \mathcal{V}_P \\ & a_{cj} + a_{pq} \leq 1, \quad \forall (e_{cj}, e_{pq}) \in \bar{\mathcal{E}} \end{aligned} \tag{60}$$

where the subscripts p and q mean the pursuit coalition P_p and evader E_q, respectively, $a_{cj} = 1$ indicates the allocation of pursuit coalition P_c to capture evader E_j, and $a_{cj} = 0$ means no assignment (similar for a_{pq}), and $z : \mathcal{E} \times \{0,1\} \to \mathbb{R}$ evaluates the assignment. By the definition of $\mathcal{C}$, the last constraint in (60) implies that for every edge in $\bar{\mathcal{E}}$, at most one associated assignment can be taken, which ensures that a pursuer does not get multiple assignments. Let $L = 1 + \max_{P_c \in \mathcal{V}_P, E_j \in \mathcal{V}_E} \varrho(X_{cj}; f_c)$ be the maximum safe distance among players plus one.

Theorem 5 (Task allocation). *For the BIP* (60)*,*

(1) the complexity is NP-hard;
(2) if $z(a_{cj}) = 0$, the solution corresponds to the most captured evaders.
(3) if $z(a_{cj}) = a_{cj}\varrho(X_{cj}; f_c)/(\min\{N_p, N_e\}L)$, the solution corresponds to the most captured evaders with the maximum sum of safe distances;
(4) if $z(a_{cj}) = -a_{cj}\varrho(X_{cj}; f_c)/(\min\{N_p, N_e\}L)$, the solution corresponds to the most captured evaders with the minimum sum of safe distances.

Proof. Regarding (1), the identical argument to the proof of Theorem 4.1 in Yan et al. (2022) proves that the well-known NP-complete *3-dimensional matching problem* (Karp, 1972) is polynomially reduced to special instances of the BIP (60). Regarding (2), it follows from the definition. Regarding (3), since $0 \leq \sum_{e_{cj} \in \mathcal{E}} z(a_{cj}) < \frac{\sum_{e_{cj} \in \mathcal{E}} a_{cj} L}{\min\{N_p, N_e\} L} \leq 1$, then $\sum_{e_{cj} \in \mathcal{E}} z(a_{cj})$ is strictly dominated by the value increment of matching one more evader. Thus, the conclusion follows directly, and (4) can be proved similarly. □

To solve (60), many solvers can be used (e.g., Gurobi, Matlab). If the number of players is large, the Sequential Matching Algorithm (Yan et al., 2022) is a 1/2 approximation polynomial algorithm, and an exact algorithm if the solution does not contain pursuit coalitions with two pursuers. Motivated by Antonyshyn et al. (2023); Yan et al. (2022), we combine the ERP winning (motion planning) with the task allocation (task planning) in a receding-horizon manner and thus propose a multiplayer receding-horizon ERP strategy (Algorithm 1) that ensures a monotonically increasing number of guaranteed captured evaders. Since at most two pursuers are needed for the ERP winning by the coalition reduction, the number of tasks to be considered is reduced from $(2^{N_p} - 1)N_e$ to $(N_p + 1)/N_p N_e$, which however, as Theorem 5 states, is still an NP-hard problem. $ERP\_Winning(X_{cj}, f_c)$ determines whether P_c guarantees an ERP winning against E_j from X_{cj}. This can be verified using the proposed steer-to-ERP approach with the positional PEFs $\{f_i^{\text{ps}}\}_{P_i \in \mathscr{P}}$, i.e, check the winning parameters and safe distances in Theorems 3 or 4.

Algorithm 1 Multiplayer ERP strategy

Initialize: $\{\boldsymbol{x}_{P_i}, \theta_{P_i}\}_{P_i \in \mathscr{P}}$, $\{\boldsymbol{x}_{E_j}\}_{E_j \in \mathscr{E}}$, PEFs $\{f_i\}_{P_i \in \mathscr{P}}$

$\mathcal{V}_P \leftarrow \{P_c \in 2^{\mathscr{P}} \mid 1 \leq |P_c| \leq 2\}$, $\mathcal{V}_E \leftarrow \mathscr{E}$
repeat
  for $P_c \in \mathcal{V}_P$, $E_j \in \mathcal{V}_E$ **do**
    Add e_{cj} to $\mathcal{E}$ if $ERP\_Winning(X_{cj}, f_c)$ is true
  $G \leftarrow (\mathcal{V}_P \cup \mathcal{V}_E, \mathcal{E})$
  $M \leftarrow$ solve the BIP (60)
  Adopt ERP winning strategy (19) or (46) for P_c if $(P_c, E_j) \in M$ for some E_j
  Adopt some (any) strategy for $E_j \in \mathcal{V}_E$ and unmatched $P_i \in \mathscr{P}$
  Update $\boldsymbol{x}_{P_i}, \theta_{P_i}, \boldsymbol{x}_{E_j}$ with a time step Δ
  Remove captured or arriving evaders from $\mathcal{V}_E$
until $\mathcal{V}_E = \emptyset$

7 Simulations

We run the Homicidal Chauffeur reach-avoid differential games in various scenarios with different team sizes and initial configurations to illustrate the theoretical results. The positional PEF (12) is used for the ERP winning.

Case 1: one pursuer P_1 and one evader E_1. We consider the parameters $\alpha_{11} = 5, r_1 = 7.31$ and $\kappa_1 = 1.5$ which satisfy the condition (16), and consider the initial states $\boldsymbol{x}_{P_1} = [-6, 7], \theta_{P_1} = -1.5$ and $\boldsymbol{x}_{E_1} = [18, 18]$ which satisfy the condition (18). By Theorem 3, the initial state is an ERP winning state and thus under the ERP winning strategy (19), P_1 is able to defend the goal region against E_1 which can take any strategy. The scenario is depicted in Fig. 5(a), where the blue dashed circle is the capture range. After the state enters $\mathcal{X}_{11}^{\text{ps}}$ in (14), the ERP winning strategy by P_1 ensures that the enclosure regions containing E_1 (in green at several time instants) never approach the goal region, and thus E_1 cannot reach the goal region before being captured.

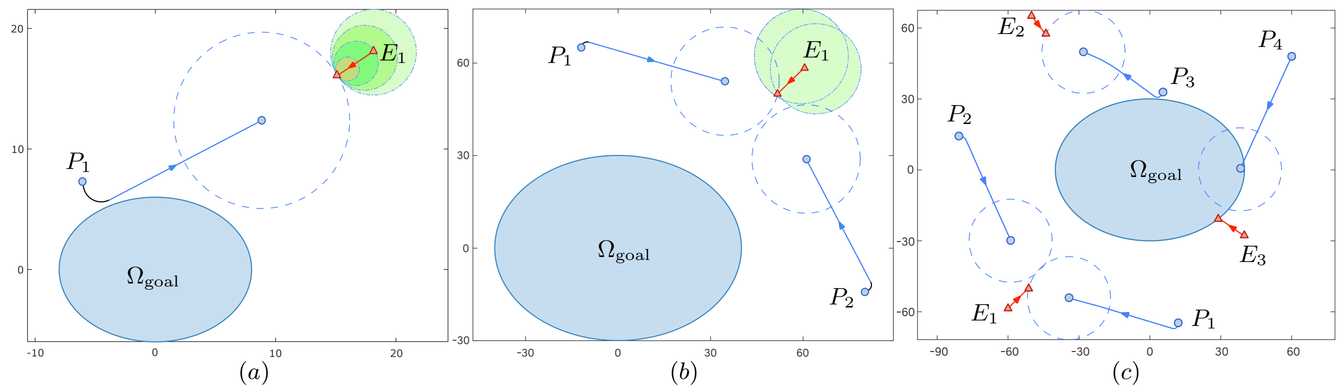

Fig. 5. Three simulations. (a) one pursuer and one evader; (b) two pursuers and one evader; (c) four pursuers and three evaders.

Case 2: two pursuers P_1, P_2 and one evader E_1. We consider $\alpha_{11} = \alpha_{21} = 4$, $r_1 = r_2 = 17.56$ and $\kappa_1 = \kappa_2 = 2$. The initial states are $\boldsymbol{x}_{P_1} = [-12, 65], \theta_{P_1} = 1.5$, $\boldsymbol{x}_{P_2} = [80, -14], \theta_{P_2} = -0.1$ and $\boldsymbol{x}_{E_1} = [60.5, 58.5]$. By Theorem 3, P_1 and P_2 cannot ensure the ERP winning against E_1 individually due to the failure of the state condition (18). However, the initial states meet the condition (2) in Theorem 4, and thus P_1 and P_2 can defend against E_1 by cooperation using the strategy (46), as in Fig. 5(b), where the enclosure regions are depicted at the instant when the state enters $\mathcal{X}_{cj}^{\mathrm{ps}}$ in (14).

Case 3: four pursuers and three evaders. The multiplayer receding-horizon ERP strategy is used in this example. The task allocation shows that the pursuit team can ensure the simultaneous ERP winning against at most two evaders, show in Fig. 5(c). More concretely, P_1 and P_2 cooperatively defend against E_1, and P_3 defends against E_2. The pursuer P_4 is tasked to pursue E_3, although it cannot ensure the ERP winning against E_3.

8 Conclusion

We presented a cooperative pursuit strategy for multiplayer Homicidal Chauffeur reach-avoid differential games in which the pursuers protect a convex region against the evaders. For the subgames, the ERP winning provides a sufficient condition to guarantee the pursuit winning without directly working with the terminal conditions. In addition to avoiding the backward analysis, the ERP winning has simple cooperation among pursuers due to the pursuit coalition reduction. The steer-to-ERP approach shows that, if a set of ERP winning states are constructed, the new ERP winning states can be generated by solving an optimization problem. The parameters, states and strategies that ensure the ERP winning with the proposed positional PEFs are able to find a part of the pursuit winning conditions. The task allocation leads to an increasing number of guaranteed captured evaders. Future work will involve two-car dynamics and distributed games. Moreover, since the task allocation at each step involves solving a combinatorial problem, for future work, we will propose heuristic methods based on players' current states to prune the pursuit coalitions that are considered for possible tasks.

References

Antonyshyn, L., Silveira, J., Givigi, S., & Marshall, J. (2023). Multiple mobile robot task and motion planning: A survey. *ACM Computing Surveys*, *55*, 1–35.

Battistini, S., & Shima, T. (2014). Differential games missile guidance with bearings-only measurements. *IEEE Transactions on Aerospace and Electronic Systems*, *50*, 2906–2915.

Bopardikar, S. D., Bullo, F., & Hespanha, J. P. (2009). A cooperative Homicidal Chauffeur game. *Automatica*, *45*, 1771–1777.

Calafiore, G. C., & Campi, M. C. (2006). The scenario approach to robust control design. *IEEE Transactions on Automatic Control*, *51*, 742–753.

Cardaliaguet, P. (1996). A differential game with two players and one target. *SIAM Journal on Control and Optimization*, *34*, 1441–1460.

Chen, M., Bansal, S., Fisac, J. F., & Tomlin, C. J. (2019). Robust sequential trajectory planning under disturbances and adversarial intruder. *IEEE Transactions on Control Systems Technology*, *27*, 1566–1582.

Chen, M., Herbert, S. L., Vashishtha, M. S., Bansal, S., & Tomlin, C. J. (2018). Decomposition of reachable sets and tubes for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, *63*, 3675–3688.

Chen, M., Zhou, Z., & Tomlin, C. J. (2017). Multiplayer reach-avoid games via pairwise outcomes. *IEEE Transactions on Automatic Control*, *62*, 1451–1457.

Elliott, R. J., & Kalton, N. J. (1972). *The existence of value in differential games*. American Mathematical Soc.

Exarchos, I., Tsiotras, P., & Pachter, M. (2015). On the suicidal pedestrian differential game. *Dynamic Games and Applications*, *5*, 297–317.

Falcone, M. (2006). Numerical methods for differential games based on partial differential equations. *International Game Theory Review*, *8*, 231–272.

Fonod, R., & Shima, T. (2018). Blinding guidance against missiles sharing bearings-only measurements. *IEEE Transactions on Aerospace and Electronic Systems*, *54*, 205–216.

Fu, H., & Liu, H. H.-T. (2023). Justification of the geometric solution of a target defense game with faster defenders and a convex target area using the HJI equation. *Automatica*, *149*, 110811.

Garcia, E., Casbeer, D. W., & Pachter, M. (2020a). Optimal strategies for a class of multi-player reach-avoid differential games in 3D space. *IEEE Robotics and Automation Letters*, *5*, 4257–4264.

Garcia, E., Casbeer, D. W., Von Moll, A., & Pachter, M. (2020b). Multiple pursuer multiple evader differential games. *IEEE Transactions on Automatic Control*, *66*, 2345–2350.

Getz, W., & Pachter, M. (1981). Two-target pursuit-evasion differential games in the plane. *Journal of Optimization Theory and Applications*, *34*, 383–403.

Isaacs, R. (1965). *Differential Games*. New York: Wiley.

Karp, R. M. (1972). Reducibility among combinatorial problems. In *Complexity of computer computations* (pp. 85–103).

Lee, Y., & Bakolas, E. (2022a). Guarding a convex target set from an attacker in Euclidean spaces. *IEEE Control Systems Letters*, *6*, 1706–1711.

Lee, Y., & Bakolas, E. (2022b). Solutions for games of guarding an arbitrary convex target set with multiple defenders in R^n. *TechRxiv: dx.doi.org/10.36227 /techrxiv.20468970.v1*, .

Lewin, J., & Breakwell, J. (1975). The surveillance-evasion game of degree. *Journal of Optimization Theory and Applications*, *16*, 339–353.

Lewin, J., & Olsder, G. (1979). Conic surveillance evasion. *Journal of Optimization Theory and Applications*, *27*, 107–125.

Li, D., Cruz, J. B., Chen, G., Kwan, C., & Chang, M.-H. (2005). A hierarchical approach to multi-player pursuit-evasion differential games. In *Proceedings of the 44th IEEE Conference on Decision and Control* (pp. 5674–5679).

Makkapati, V. R., & Tsiotras, P. (2019). Optimal evading strategies and task allocation in multi-player pursuit–evasion problems. *Dynamic Games and Applications*, *9*, 1168–1187.

Margellos, K., & Lygeros, J. (2011). Hamilton–Jacobi formulation for reach–avoid differential games. *IEEE Transactions on Automatic Control*, *56*, 1849–1861.

Merz, A. W. (1971). *The homicidal chauffeur–a differential game*. Ph.D. thesis Stanford University.

Merz, A. W. (1974). The homicidal chauffeur. *AIAA Journal*, *12*, 259–260.

Mitchell, I. M. (2002). *Application of level set methods to control and reachability problems in continuous and hybrid systems*. Ph.D. thesis Stanford University.

Mitchell, I. M., Bayen, A. M., & Tomlin, C. J. (2005). A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games. *IEEE Transactions on Automatic Control*, *50*, 947–957.

Mohanan, J., Kothuri, N., & Bhikkaji, B. (2020). The target guarding problem: A real time solution for noise corrupted measurements. *European Journal of Control*, *54*, 111–118.

Oyler, D. W., Kabamba, P. T., & Girard, A. R. (2016). Pursuit–evasion games in the presence of obstacles. *Automatica*, *65*, 1–11.

Pachter, M., & Yavin, Y. (1981). A stochastic homicidal chauffeur pursuit-evasion differential game. *Journal of Optimization Theory and Applications*, *34*, 405–424.

Patsko, V. S., & Turova, V. L. (2011). Homicidal Chauffeur game: History and modern studies. In *Advances in Dynamic Games: Theory, Applications, and Numerical Methods for Differential and Stochastic Games* (pp. 227–251).

Pihl, M. J., Marcher, J., & Jensen, J. A. (2012). Phased-array vector velocity estimation using transverse oscillations. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, *59*, 2662–2675.

Shishika, D., & Kumar, V. (2018). Local-game decomposition for multiplayer perimeter-defense problem. In *IEEE Conference on Decision and Control* (pp. 2093–2100).

Shishika, D., & Kumar, V. (2020). A review of multi agent perimeter defense games. In *International Conference on Decision and Game Theory for Security* (pp. 472–485).

Soumekh, M. (1997). Phased array imaging of moving targets with randomized beam steering and area spotlighting. *IEEE Transactions on Image Processing*, *6*, 736–749.

Von Moll, A., Garcia, E., Casbeer, D., Suresh, M., & Swar, S. C. (2020). Multiple-pursuer, single-evader border defense differential game. *Journal of Aerospace Information Systems*, *17*, 407–416.

Von Moll, A., Pachter, M., Shishika, D., & Fuchs, Z. (2022). Circular target defense differential games*. *IEEE Transactions on Automatic Control*, *1*, 1–14.

Wasz, P., Pachter, M., & Pham, K. (2019). Two-an-one pursuit with a non-zero capture radius. In *2019 27th Mediterranean Conference on Control and Automation (MED)* (pp. 577–582).

Yan, R., Deng, R., Duan, X., Shi, Z., & Zhong, Y. (2023a). Multiplayer reach-avoid differential games with simple motions: A review. *Frontiers in Control Engineering*, *3*.

Yan, R., Deng, R., Lai, H., Zhang, W., Shi, Z., & Zhong, Y. (2023b). Homicidal Chauffeur reach-avoid games via guaranteed winning strategies. *IEEE Transactions on Automatic Control*, (pp. 1–16).

Yan, R., Duan, X., Shi, Z., Zhong, Y., & Bullo, F. (2022). Matching-based capture strategies for 3D heterogeneous multiplayer reach-avoid differential games. *Automatica*, *140*, 110207.

Yan, R., Shi, Z., & Zhong, Y. (2019). Reach-avoid games with two defenders and one attacker: An analytical approach. *IEEE Transactions on Cybernetics*, *49*, 1035–1046.

Yan, R., Shi, Z., & Zhong, Y. (2020). Guarding a subspace in high-dimensional space with two defenders and one attacker. *IEEE Transactions on Cybernetics*, *52*, 3998–4011.

Yan, R., Shi, Z., & Zhong, Y. (2020). Task assignment for multiplayer reach-avoid games in convex domains via analytical barriers. *IEEE Transactions on Robotics*, *36*, 107–124.

Yan, R., Shi, Z., & Zhong, Y. (2021). Optimal strategies for the lifeline differential game with limited lifetime. *International Journal of Control*, *94*, 2238–2251.

Zhou, Z., Takei, R., Huang, H., & Tomlin, C. J. (2012). A general, open-loop formulation for reach-avoid games. In *IEEE Conference on Decision and Control* (pp. 6501–6506).

Zhou, Z., Zhang, W., Ding, J., Huang, H., Stipanović, D. M., & Tomlin, C. J. (2016). Cooperative pursuit with voronoi partitions. *Automatica*, *72*, 64–72.