

# The Complexity of Learning Temporal Properties

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## Abstract

We consider the problem of learning temporal logic formulas from examples of system behavior. Learning temporal properties has crystallized as an effective mean to explain complex temporal behaviors. Several efficient algorithms have been designed for learning temporal formulas. However, the theoretical understanding of the complexity of the learning decision problems remains largely unexplored. To address this, we study the complexity of the passive learning problems of three prominent temporal logics, Linear Temporal Logic (LTL), Computation Tree Logic (CTL) and Alternating-time Temporal Logic (ATL) and several of their fragments. We show that learning formulas using an unbounded amount of occurrences of binary operators is NP-complete for all of these logics. On the other hand, when investigating the complexity of learning formulas with bounded amount of occurrences of binary operators, we exhibit discrepancies between the complexity of learning LTL, CTL and ATL formulas (with a varying number of agents).

## 1 Introduction

Temporal logics are the de-facto standard for expressing temporal properties for software and cyber-physical systems. Originally introduced in the context of program verification [28, 14], temporal logics are now well-established in numerous areas, including reinforcement learning [37, 22, 9], motion planning [15, 11], process mining [12], and countless others. The popularity of temporal logics can be attributed to their unique blend of mathematical rigor and resemblance to natural language.

Until recently, formulating properties in temporal logics has been a manual task, requiring human intuition and expertise [6, 36]. To circumvent this step, in the past ten years, there have been numerous works to automatically learn (i.e., generate) properties in temporal logic. Among them, a substantial number of works [26, 10, 31, 23, 38] target Linear Temporal Logic (LTL) [28]. There is now a growing interest [35, 30] in learning formulas in Computation Tree Logic (CTL) [14] and Alternating-time Temporal Logic (ATL) [1] due to their ability to express branching-time properties of multi-agent systems.

While existing approaches for learning temporal properties demonstrate impressive empirical performance, their computational complexity remains largely unexplored. The only noteworthy related works are [16] and the follow-up work [24]. They present NP-completeness results for learning formulas in LTL and several of its fragments.

In this work, we extend the existing results to encompass a wider range of LTL operators. Moreover, we extend the study to learning CTL and ATL formulas.

	Unbounded use of binary operators	Bounded use of binary operators	
		$\mathbf{X} \in \mathbf{U}^t$	$\mathbf{X} \notin \mathbf{U}^t$
			$\{\mathbf{F}, \mathbf{G}\} \subseteq \mathbf{U}^t$
LTL	NP-c	L	
CTL		NP-c	NL-c
ATL(2)		NP-c	P-c
ATL( $p$ )		NP-c	

Table 1: Summary of the complexity results for learning LTL, CTL and ATL.  $\text{ATL}(k)$  corresponds to learning ATL with  $k$  agents, while ATL refers to learning ATL with the set of agents as input.  $\mathbf{U}^t$  refers to the set of unary operators allowed.

To elaborate on our contributions, we describe the precise problem that we consider, the fundamental *passive learning* problem [17]. Its decision version asks the following question: given two sets  $\mathcal{P}, \mathcal{N}$  of positive and negative examples of system behavior and a size bound  $B$ , does there exist a formula of size at most  $B$  satisfied by the positive examples and violated by the negative examples.

Our instantiation of the above problem varies slightly depending on the considered logic. Indeed, LTL-formulas express linear-time properties, CTL-formulas express branching-time properties, and ATL-formulas express properties involving on multi-agent systems. Accordingly, the input examples for learning LTL, CTL and ATL are linear structures (equivalently infinite words), Kripke structures and concurrent game structures, respectively. We refer to Section 2 for formal definitions and other prerequisites,

**We summarize our contributions in Table 1.** Our first result, illustrated in the left column, shows that without any restriction on the use of binary operators, the learning problem for any logic is NP-complete, regardless of the binary operators allowed. The NP-hardness results are (heavily) inspired by the proofs by [24] and (mostly) use reductions from the hitting set problem—one of Karp’s 21 NP-complete problem. The details of the proofs are given in Section 3.

In the search for logic fragments with lower complexities, we turn to formulas using only a bounded amount of binary operators, and unary operators in a set  $\mathbf{U}^t \subseteq \{\neg, \mathbf{X}, \mathbf{F}, \mathbf{G}\}$ , depicted in the middle column. Note that the bound on the number of binary operators is fixed beforehand (i.e. it is part of the learning problem itself, not of the input). In this case, the complexity of the learning problems varies between different logics and unary operators. Importantly, we exhibit fragments where the learning problem is decidable in polynomial time. This is handled in Section 4.

Note that this work largely extends a preliminary version [8].

**Related Works.** The closest related works are [16] and [24]. Both works consider learning problems in several fragments of LTL, especially involving boolean operators such as  $\vee$  and  $\wedge$ , and temporal operators such as  $\mathbf{X}, \mathbf{F}$  and  $\mathbf{G}$  and prove their NP-completeness. We extend part of their work by categorizing fragments based on the arity of the operators and studying which type of operators contribute to the hardness. Moreover, there are several differences in the parameters considered for the learning problem. For instance, the above works consider the size upper bound  $B$  to be in binary, while we assume  $B$  given in unary. Considering size bound in unary is often justified since one may want to output a unary-sized formula anyway. We discuss more thoroughly such differences in Section 3.1.1. Nonetheless, in addition to LTL, we widen the scope of the complexity results to CTL and ATL.

In the past, complexity analysis of passive learning has been studied for formalisms other than temporal logics. For instance, [18] and [2] proved NP-completeness of the passive learning problems of deterministic finite automata (DFAs) and regular expressions (REs).

When considering temporal logics, most related works focus on devising efficient algorithms for learning temporal logic. Several works learn LTL (or its important fragments) by either exploiting constraint solving [26, 10, 33] or efficient enumerative search [31, 38]. Some recent works rely on neuro-symbolic approaches to learn LTL formulas from noisy data [23, 39]. For CTL, many works resort to handcrafted templates [13, 40] for simple enumerative search, while others learn formulas of arbitrary structure through constraint solving [35, 30].

There are also works on learning other logics such as Signal Temporal Logic [7, 25], Metric Temporal Logic [32], Past LTL [3], Property Specification Language [34], etc.

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## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Definitions</b>	<b>4</b>
2.1	LTL: syntax and semantics	4
2.2	ATL and CTL: syntax and semantics	6
2.3	Decision problems	8
<b>3</b>	<b>Learning with non-unary binary operators is NP-hard</b>	<b>9</b>
3.1	LTL learning	9
3.1.1	What is done in [24]	9
3.2	Our results	11
3.2.1	Two useful lemmas	12
3.2.2	Proof of Theorem 10: when $B^1 \cap \{\vee, \Rightarrow, \Leftarrow, \wedge, \neg\Rightarrow, \neg\Leftarrow\} \neq \emptyset$	13
3.2.3	Proof of Theorem 10: when $B^1 \cap \{\neg\vee, \neg\wedge\} \neq \emptyset$	15
3.2.4	Proof of Theorem 10: with the temporal operators <b>W</b> and <b>M</b>	18
3.2.5	Proof of Theorem 10: when $B^1 \cap \{\Leftrightarrow, \neg\Leftrightarrow\} \neq \emptyset$	19
3.2.6	Proof of Theorem 10: with the temporal operators <b>U</b> and <b>R</b>	23
3.3	CTL learning is at least as hard as LTL learning	26
<b>4</b>	<b>Learning formulas using only unary operators</b>	<b>29</b>
4.1	LTL learning	30
4.2	Abstract recipe for NP-hardness proofs and binary operators	32
4.2.1	Abstract recipe	34
4.2.2	Handling binary operators	34
4.3	CTL learning	42
4.3.1	With the next operator <b>X</b>	42
4.3.2	Without the next operator <b>X</b>	48
4.4	ATL learning without the operator <b>X</b>	56
4.4.1	Alternating ATL-formulas and turn-based structures	56
4.4.2	ATL learning with two agents and operators <b>F</b> and <b>G</b>	59

4.4.3	ATL learning with two agents and with one of the two operators <b>F</b> and <b>G</b>	66
4.4.4	ATL learning with three agents	69

## 5 Conclusion and future Work

73

## 2 Definitions

**Complexity classes** In this paper, we are going to show several completeness results. As can be seen in Table 1, the complexity classes that we will consider are L (logspace), NL (non-deterministic logspace), P (polynomial time), and NP (non-deterministic polynomial time). Note that all the reductions that we will define are logspace reductions.

**Some notations** We let  $\mathbb{N}$  denote the set of all integers and  $\mathbb{N}_1$  denote the set of all positive integers. Furthermore, for all  $i \leq j \in \mathbb{N}$ , we let  $[i, \dots, j] \subseteq \mathbb{N}$  denote the set of integers  $\{i, i+1, \dots, j\}$ .

Given any non-empty set  $Q$ , we let  $Q^*$ ,  $Q^+$  and  $Q^\omega$  denote the sets of finite, non-empty finite and infinite sequences of elements in  $Q$ , respectively. For all  $\rho \in Q^+$ , we denote by  $|\rho| \in \mathbb{N}$  the length of  $\rho$ , i.e. its number of elements.

Furthermore, for all  $\bullet \in \{+, \omega\}$ ,  $\rho \in Q^\bullet$  and  $i \in \mathbb{N}_1$ , if  $\rho$  has at least  $i$  elements, we let:

- $\rho[i] \in Q$  denote the  $i$ -th element in  $\rho$ , in particular  $\rho[1] \in Q$  denotes the first element of  $\rho$ ;
- $\rho[:i] \in Q^+$  denote the non-empty finite sequence  $\rho_1 \cdots \rho_i \in Q^+$ ;
- $\rho[i:] \in Q^\bullet$  denote the non-empty sequence  $\rho_i \cdot \rho_{i+1} \cdots \in Q^\bullet$ , in particular we have  $\rho[1:] = \rho$ .

For the remainder of this section, we fix a non-empty set of propositions  $\text{Prop}$ .

### 2.1 LTL: syntax and semantics

Before introducing LTL-formulas [29], let us first introduce the objects on which these formulas will be interpreted: infinite words.

**Infinite and ultimately periodic words.** Given a set of propositions  $\text{Prop}$ , an infinite word is an element of the set  $(2^{\text{Prop}})^\omega$ . Furthermore, we will be particularly interested in ultimately periodic words, and in size-1 ultimately periodic words. They are formally defined below.

**Definition 1.** Consider a set of propositions  $\text{Prop}$ . An ultimately periodic word  $w \in (2^{\text{Prop}})^\omega$  is such that  $w = u \cdot v^\omega$  for some finite words  $u \in (2^{\text{Prop}})^*$ ,  $v \in (2^{\text{Prop}})^+$ . In that case, we set the size  $|w|$  of  $w$  to be equal to  $|w| := |u| + |v|$ . Then, for all sets  $S$  of ultimately periodic words, we set  $|S| := \sum_{w \in S} |w|$ .

An ultimately periodic word  $w$  is of size-1 if  $|w| = 1$ . That is,  $w = \alpha^\omega$  for some  $\alpha \in 2^{\text{Prop}}$ .

**Set of operators** The LTL, CTL and ATL-formulas that we will consider in the following will use the following temporal operators: **X** (neXt), **F** (Future), **G** (Globally), **U** (Until), **R** (Release), **W** (Weak until), **M** (Mighty release).

We let  $\text{Op}_{\text{Un}} := \{\neg, \mathbf{X}, \mathbf{F}, \mathbf{G}\}$  and  $\text{Op}_{\text{Bin}}^{\text{tp}} := \{\mathbf{U}, \mathbf{R}, \mathbf{W}, \mathbf{M}\}$  denote the sets of unary and binary operators respectively. We also let  $\text{Op}_{\text{Bin}}^{\text{lg}}$  denote the set of all logical binary operators, i.e. the classical operators along with their negations:  $\text{Op}_{\text{Bin}}^{\text{lg}} := \{\vee, \wedge, \Rightarrow, \Leftarrow, \Leftrightarrow, \neg\vee, \neg\wedge, \neg\Rightarrow, \neg\Leftarrow, \neg\Leftrightarrow\}$ .

**Syntax.** For all  $U^t \subseteq \text{Op}_{\text{Un}}$ ,  $B^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$  and  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$ , we denote by  $\text{LTL}(\text{Prop}, U^t, B^t, B^l)$  the set of LTL-formulas defined inductively as follows:

$$\varphi ::= p \mid \neg\varphi \mid *_1\varphi \mid \varphi *_2\varphi$$

where  $p \in \text{Prop}$ ,  $*_1 \in U^t$  and  $*_2 \in B^t \cup B^l$ .

We define the size  $|\varphi|$  of an LTL-formula  $\varphi$  to be its number of sub-formulas. That set of sub-formulas  $\text{SubF}(\varphi)$  is defined inductively as follows:

- $\text{SubF}(p) := \{p\}$  for all  $p \in \text{Prop}$ ;
- $\text{SubF}(\bullet\varphi) := \text{SubF}(\varphi) \cup \{\bullet\varphi\}$  for all unary operators  $\bullet \in \text{Op}_{\text{Un}}$ ;
- $\text{SubF}(\varphi_1 \bullet \varphi_2) := \text{SubF}(\varphi_1) \cup \text{SubF}(\varphi_2) \cup \{\varphi_1 \bullet \varphi_2\}$  for all binary operators  $\bullet \in \text{Op}_{\text{Bin}}^{\text{tp}} \cup \text{Op}_{\text{Bin}}^{\text{lg}}$ .

We also denote by  $\text{Prop}(\varphi)$  the set  $\text{Prop} \cap \text{SubF}(\varphi)$ . Furthermore, we say that a set of propositions  $Y \subseteq \text{Prop}$  *occurs* in a formula  $\varphi$  if  $Y \cap \text{Prop}(\varphi) \neq \emptyset$ . (This notation will also be used with CTL and ATL-formulas.)

Finally, we let  $|\varphi|_{\text{Bin}} \in \mathbb{N}$  denote the number of usage of binary operators in  $\varphi$ , i.e.:

$$|\varphi|_{\text{Bin}} := |\text{SubBin}(\varphi)|$$

with

$$\text{SubBin}(\varphi) := \{\varphi_1 \bullet \varphi_2 \in \text{SubF}(\varphi) \mid \varphi_1, \varphi_2 \in \text{SubF}(\varphi), \bullet \in \text{Op}_{\text{Bin}}^{\text{tp}} \cup \text{Op}_{\text{Bin}}^{\text{lg}}\}$$

**Semantics.** We define the semantics of LTL-formulas. That is, given an LTL-formula  $\varphi$  and an infinite word  $w \in (2^{\text{Prop}})^\omega$ , we define when  $w \models \varphi$ , i.e. when  $\varphi$  accepts  $w$ , (otherwise it rejects it). In order to give a semantics to all binary logical operators, when  $w \models \varphi$  is true, it is seen as the boolean value **True**, and when  $w \models \varphi$  is false, it is seen as the boolean value **False**. Then, for all  $w \in (2^{\text{Prop}})^\omega$ , we have (for  $* \in B^l$ ):

$$\begin{aligned} w \models p &\text{ iff } p \in w[0], \\ w \models \neg\varphi &\text{ iff } w \not\models \varphi; \\ w \models \varphi_1 * \varphi_2 &\text{ iff } (w \models \varphi_1) * (w \models \varphi_2) = \text{True}; \\ w \models \mathbf{X}\varphi &\text{ iff } w[2:] \models \varphi; \\ w \models \mathbf{F}\varphi &\text{ iff } \exists i \in \mathbb{N}_1, w[i:] \models \varphi; \\ w \models \mathbf{G}\varphi &\text{ iff } \forall i \in \mathbb{N}_1, w[i:] \models \varphi; \\ w \models \varphi_1 \mathbf{U} \varphi_2 &\text{ iff } \exists i \in \mathbb{N}_1, w[i:] \models \varphi_2 \text{ and } \forall 1 \leq j \leq i-1, w[j:] \models \varphi_1; \\ w \models \varphi_1 \mathbf{R} \varphi_2 &\text{ iff } w \models \neg(\neg\varphi_1 \mathbf{U} \neg\varphi_2) \\ w \models \varphi_1 \mathbf{W} \varphi_2 &\text{ iff } w \models (\varphi_1 \mathbf{U} \varphi_2) \vee \mathbf{G} \varphi_1; \\ w \models \varphi_1 \mathbf{M} \varphi_2 &\text{ iff } w \models (\varphi_1 \mathbf{R} \varphi_2) \wedge \mathbf{F} \varphi_1 \end{aligned}$$

Given two LTL-formulas  $\varphi, \varphi'$ , we write  $\varphi \implies \varphi'$  when, for all ultimately periodic words  $w$ , we have that if  $w \models \varphi$ , then  $w \models \varphi'$ . We write  $\varphi \equiv \varphi'$  if  $\varphi \implies \varphi'$  and  $\varphi' \implies \varphi$ .

Given any set  $S \subseteq (2^{\text{Prop}})^\omega$  of infinite words, we say that an LTL-formula  $\varphi$  accepts  $S$  if it accepts all words in  $S$ , and we say that  $\varphi$  rejects  $S$  if it rejects all words in  $S$ . Furthermore, such an LTL-formula  $\varphi$  distinguishes two sets of infinite words  $S \subseteq (2^{\text{Prop}})^\omega$  and  $S' \subseteq (2^{\text{Prop}})^\omega$  if it accepts  $S$  and rejects  $S'$ , or if it accepts  $S'$  and rejects  $S$ .

## 2.2 ATL and CTL: syntax and semantics

Let us first introduce the notion of concurrent game structures (CGS), on which ATL-formulas are evaluated. We then introduce Kripke structures, a special kind of concurrent game structure, on which CTL-formulas are evaluated.

**Definition 2.** A concurrent game structure (CGS for short) is a the tuple  $C = \langle Q, I, k, \text{Prop}, \pi, d, \delta \rangle$  where,

- $Q$  is the finite set of states;
- $I \subseteq Q$  is the set of initial states;
- $k \in \mathbb{N}$  denotes the number of agents, we denote by  $\text{Ag} := [1, \dots, k]$  the set of  $k$  agents;
- $\pi : Q \mapsto 2^{\text{Prop}}$  maps each state  $s \in Q$  to the set of propositions that hold in  $s$ ;
- $d : Q \times \text{Ag} \rightarrow \mathbb{N}^+$  maps each state and agent to the number of actions available to that agent at that state;
- $\delta : Q_{\text{Act}} \rightarrow Q$  is the function mapping every state and tuple of one action per agent to the next state, where  $Q_{\text{Act}} := \{(q, \alpha_1, \dots, \alpha_k) \mid q \in Q, \forall a \in \text{Ag}, \alpha_a \in [1, \dots, d(q, a)]\}$ .

For all states  $q \in Q$  and coalitions of agents  $A \subseteq \text{Ag}$ , we let  $\text{Act}_A(q) := \{\alpha = (\alpha_a)_{a \in A} \mid \forall a \in \text{Ag}, \alpha_a \in [1, \dots, d(q, a)]\}$ . Then, for all tuple  $\alpha = (\alpha_a)_{a \in A} \in \text{Act}_A(q)$  of one action per agent in  $A$ , we let:

$$\text{Succ}(q, \alpha) := \{q' \in Q \mid \exists \alpha' = (\alpha'_a)_{a \in \text{Ag} \setminus A} \in \text{Act}_{\text{Ag} \setminus A}(q), \delta(q, (\alpha, \alpha')) = q'\}$$

Finally, the size  $|C|$  of the concurrent structure  $C$  is equal to:  $|C| = |Q_{\text{Act}}| + |\text{Prop}|$ .

Unless otherwise stated, a concurrent game structure  $C$  will always refer to the tuple  $\langle Q, I, k, \text{Prop}, \pi, d, \delta \rangle$ .

A Kripke structure is then a concurrent game structure where there is only one agent.

**Definition 3.** A Kripke structure is a concurrent game structure  $C$  where  $k = 1$  and  $d$  and  $\delta$  are replaced by subsets of successor states  $\emptyset \neq \text{Succ}(q) \subseteq Q$  for all states  $q \in Q$ .

Unless otherwise state, a Kripke structure  $C$  will will always refer to the tuple  $\langle Q, I, k, \text{Prop}, \pi, \text{Succ} \rangle$ .

In a concurrent game structure, a strategy for an agent is a function that prescribes to the agent what to do as a function of the history of the game, i.e. of the finite sequence of states seen so far. Furthermore, given a coalition of agents and a tuple of one strategy per agent in the coalition, we define the set of infinite sequences of states that can occur with this tuple of strategies, from any state. This is formally defined below.

**Definition 4.** Consider a concurrent game structure  $C$  and a agent  $a \in [1, \dots, k]$ . A strategy for agent  $a$  is a function  $s_a : Q^+ \rightarrow \mathbb{N}_1$  such that, for all  $\rho = \rho_0 \dots \rho_n \in Q^+$ , we have  $s_a(\rho) \leq d(\rho_n, a)$ . We denote by  $\text{S}_a$  the set of strategies available to Agent  $a \in [1, \dots, k]$ .

Given any coalition (or subset) of agents  $A \subseteq [1, \dots, k]$ , a strategy profile for the coalition  $A$  is a tuple  $s = (s_a)_{a \in A}$  of one strategy per agent in  $A$ . We denote by  $\text{S}_A$  the set of strategy profiles for the coalition  $A$ . Given any such strategy profile  $s$ , for all states  $q \in Q$ , we let  $\text{Out}^Q(q, s) \subseteq Q^\omega$  denote the set of infinite paths  $\rho$  that are compatible with the strategy profile  $s$  from  $q$ , i.e.:

$$\text{Out}^Q(q, s) := \{\rho \in Q^\omega \mid \rho[1] = q, \forall i \in \mathbb{N}_1, \rho[i+1] \in \text{Succ}(\rho[i], (s_a(\rho[:i]))_{a \in A})\}$$

In a Kripke structure, we simply consider the set of all infinite paths  $\text{Out}^Q(q)$  that can occur from a specific state  $q \in Q$ , regardless of strategies:

$$\text{Out}^Q(q) := \{\rho \in Q^\omega \mid \rho[1] = q, \forall i \in \mathbb{N}_1, \rho[i+1] \in \text{Succ}(\rho[i])\}$$

**Syntax.** To define the syntax of ATL and CTL-formulas, we introduce two types of formulas: state formulas and path formulas. Intuitively, state formulas express properties of states, where the strategic quantifier occurs, whereas path formulas express temporal properties of paths. For ease of notation, we denote state formulas and path formulas with the Greek letter  $\phi$  and the Greek letter  $\psi$ , respectively. Consider some  $U^t \subseteq \text{Op}_{U_n}$ ,  $B^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$  and  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $k \in \mathbb{N}_1$ . Then, we denote by  $\text{ATL}^k(\text{Prop}, U^t, B^t, B^l)$  the set of ATL-state formulas defined by the grammar:

$$\phi ::= p \mid \neg\phi \mid \phi * \phi \mid \langle\langle A \rangle\rangle\psi$$

where  $p \in \text{Prop}$ ,  $*$   $\in B^l$ ,  $A \subseteq \text{Ag} := \{1, \dots, k\}$  is a subset of agents, and  $\psi$  is an  $\text{ATL}^k(\text{Prop}, U^t, B^t, B^l)$ -path formula. Note that, for CTL-formulas, we have  $k = 1$ . Hence, there are only two possible subsets:  $\emptyset$  and  $\text{Ag}$  itself. Usually, in the CTL syntax,  $\langle\langle \emptyset \rangle\rangle\psi$  is denoted  $\forall\psi$  whereas  $\langle\langle \text{Ag} \rangle\rangle\psi$  is denoted  $\exists\psi$ .

Next,  $\text{ATL}^k(\text{Prop}, U^t, B^t, B^l)$ -path formulas are given by the grammar

$$\psi ::= *_1\phi \mid \phi *_2\phi$$

where  $*_1 \in U^t \setminus \{-\}$  and  $*_2 \in B^t$ . We denote by  $\text{ATL}^k$  the set of all  $\text{ATL}^k$ -formulas, with CTL referring to  $\text{ATL}^1$ . The set of sub-formulas  $\text{SubF}(\phi)$  of a formula  $\phi$  is then defined inductively as follows:

- $\text{SubF}(p) := \{p\}$  for all  $p \in \text{Prop}$ ;
- $\text{SubF}(\neg\phi) := \{\neg\phi\} \cup \text{SubF}(\phi)$ ;
- $\text{SubF}(\langle\langle A \rangle\rangle(\bullet\phi)) := \{\langle\langle A \rangle\rangle(\bullet\phi)\} \cup \text{SubF}(\phi)$  for all  $\bullet \in \text{Op}_{U_n}$  and  $A \subseteq \text{Ag}$ ;
- $\text{SubF}(\phi_1 \bullet \phi_2) := \{\phi_1 \bullet \phi_2\} \cup \text{SubF}(\phi_1) \cup \text{SubF}(\phi_2)$  for all  $\bullet \in \text{Op}_{\text{Bin}}^{\text{lg}}$ ;
- $\text{SubF}(\langle\langle A \rangle\rangle(\phi_1 \bullet \phi_2)) := \{\langle\langle A \rangle\rangle(\phi_1 \bullet \phi_2)\} \cup \text{SubF}(\phi_1) \cup \text{SubF}(\phi_2)$  for all  $\bullet \in \text{Op}_{\text{Bin}}^{\text{tp}}$  and  $A \subseteq \text{Ag}$ .

The size  $|\phi|$  of a ATL-formula is defined as its number of sub-formulas:  $|\phi| := |\text{SubF}(\phi)|$ . Finally, we also let  $\text{NbBin}(\phi) \in \mathbb{N}$  denote the number of usage of binary operators in  $\phi$ , i.e.:

$$|\phi|_{\text{Bin}} := |\text{SubBin}(\phi)|$$

with

$$\text{SubBin}(\phi) := \{\phi_1 \bullet \phi_2 \in \text{SubF}(\phi) \mid \phi_1, \phi_2 \in \text{SubF}(\phi), \bullet \in \text{Op}_{\text{Bin}}^{\text{tp}} \cup \text{Op}_{\text{Bin}}^{\text{lg}}\}$$

**Semantics.** As mentioned above, we interpret  $\text{ATL}^k$ -formulas over CGS with the set of agents  $\text{Ag} := \{1, \dots, k\}$  using the standard definitions [1]. Given a state  $s$  and a state formula  $\phi$ , we define when  $\phi$  holds in state  $s$ , denoted using  $s \models \phi$ , inductively as follows:

$$\begin{aligned} s \models p & \text{ iff } p \in \pi(s), \\ s \models \neg\phi & \text{ iff } s \not\models \phi, \\ s \models \phi_1 * \phi_2 & \text{ iff } (s \models \phi_1) * (s \models \phi_2) = \text{True}, \\ s \models \langle\langle A \rangle\rangle\psi & \text{ iff } \exists s \in S_A, \forall \pi \in \text{Out}^Q(q, s), \pi \models \psi \end{aligned}$$

where  $*$   $\in$   $\text{Op}_{\text{Bin}}^{\text{lg}}$ . In a Kripke structure, this last line can be rewritten as follows (where the first line corresponds to  $A = \emptyset$  and the second to  $A = \text{Ag}$ ):

$$\begin{aligned} s \models \exists \psi \text{ iff } \exists \pi \in \text{Out}^Q(q), \pi \models \psi \\ s \models \forall \psi \text{ iff } \forall \pi \in \text{Out}^Q(q), \pi \models \psi \end{aligned}$$

Furthermore, given a path  $\pi$  and a path formula  $\psi$ , we define when  $\psi$  holds for the path  $\pi$ , also denoted using  $\pi \models \phi$ , inductively as follows:

$$\begin{aligned} \pi \models \mathbf{X} \phi \text{ iff } \pi[2:] \models \phi; \\ \pi \models \mathbf{F} \phi \text{ iff } \exists i \in \mathbb{N}_1, \pi[i:] \models \phi; \\ \pi \models \mathbf{G} \phi \text{ iff } \forall i \in \mathbb{N}_1, \pi[i:] \models \phi; \\ \pi \models \phi_1 \mathbf{U} \phi_2 \text{ iff } \exists i \in \mathbb{N}_1, \pi[i:] \models \phi_2 \text{ and } \forall 1 \leq j \leq i-1, \pi[j:] \models \phi_1; \\ \pi \models \phi_1 \mathbf{R} \phi_2 \text{ iff } \pi \models \neg(\neg\phi_1 \mathbf{U} \neg\phi_2) \\ \pi \models \phi_1 \mathbf{W} \phi_2 \text{ iff } \pi \models (\phi_1 \mathbf{U} \phi_2) \vee \mathbf{G} \phi_1; \\ \pi \models \phi_1 \mathbf{M} \phi_2 \text{ iff } \pi \models (\phi_1 \mathbf{R} \phi_2) \wedge \mathbf{F} \phi_1 \end{aligned}$$

We now say that an ATL-formula  $\phi$  holds on a CGS  $C$ , denoted by  $C \models \phi$ , if  $s \models \phi$  for all initial states  $s \in I$  of  $C$ .

We use the notations  $\implies$  and  $\equiv$  as for LTL-formulas.

### 2.3 Decision problems

We define the LTL, CTL and  $\text{ATL}^k$  learning problems below (for  $k \in \mathbb{N}_1$ ), where a model for LTL is an ultimately periodic word, a model for CTL is a Kripke structure and a model for  $\text{ATL}^1$  is a concurrent game structure on the set of agents  $\text{Ag} := \{1, \dots, k\}$ .

**Definition 5.** Let  $L \in \{\text{LTL}, \text{CTL}, \text{ATL}^k \mid k \in \mathbb{N}_1\}$  and consider some sets of operators  $U^t \subseteq \text{Op}_{\text{Un}}$ ,  $B^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$  and  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$ . For  $n \in \mathbb{N} \cup \{\infty\}$ , we denote by  $L_{\text{Learn}}(U^t, B^t, B^l, n)$  the following decision problem:

- *Input:*  $(\text{Prop}, \mathcal{P}, \mathcal{N}, B)$  where  $\text{Prop}$  is a set of propositions,  $\mathcal{P}, \mathcal{N}$  are two finite sets of models for  $L$ , and  $B \in \mathbb{N}$ .
- *Output:* yes iff there exists an  $L$ -formula  $\varphi \in L(\text{Prop}, U^t, B^t, B^l)$  such that  $|\varphi| \leq B$ ,  $|\varphi|_{\text{Bin}} \leq n$  and  $\varphi$  separates  $\mathcal{P}$  and  $\mathcal{N}$ , i.e. such that:
  - for all  $X \in \mathcal{P}$ , we have  $X \models \varphi$ ;
  - for all  $X \in \mathcal{N}$ , we have  $X \not\models \varphi$ .

The size of the input is equal to  $|\text{Prop}| + |\mathcal{P}| + |\mathcal{N}| + B$  (i.e.  $B$  is written in unary).

As mentioned in the introduction), since the model checking problems for LTL, CTL and ATL can be decided in polynomial time [1], the problems  $L_{\text{Learn}}$ ,  $\text{CTL}_{\text{Learn}}$  and  $\text{ATL}_{\text{Learn}}^k$  are all in NP, using a straightforward guess-and-check subroutine.

**Proposition 6.** For all  $U^t \subseteq \text{Op}_{\text{Un}}$ ,  $B^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$ ,  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N} \cup \{\infty\}$ , the decision problems  $L_{\text{Learn}}(U^t, B^t, B^l)$ ,  $\text{CTL}_{\text{Learn}}(U^t, B^t, B^l)$ , and  $\text{ATL}_{\text{Learn}}^k(U^t, B^t, B^l, k)$  for all  $k \in \mathbb{N}_1$  are in NP.



### 3 Learning with non-unary binary operators is NP-hard

In this section, we study the complexity of the learning decision problems in the case where the number of occurrence of binary operators is unbounded. We first show that, in this setting, LTL learning is NP-hard. We then show that CTL learning is at least as hard as LTL learning, which, in turn, implies that CTL learning with unbounded occurrence of binary operators is NP-hard. This holds as well for  $\text{ATL}^k$  learning, for all  $k \in \mathbb{N}$ .

#### 3.1 LTL learning

Let us consider LTL learning. Before we proceed to our contributions, let us discuss the very important related work [24].

##### 3.1.1 What is done in [24]

In [24], the authors study LTL learning. However, the setting that they consider differs in several ways with the setting that we consider in this paper. We list the main differences below.

- The letters of the words considered in [24] are propositions (i.e. elements of  $\text{Prop}$ ) whereas the letters that we consider are subsets of propositions (i.e. elements of  $2^{\text{Prop}}$ ).
- The words that we consider are infinite whereas the words considered in [24] are finite.
- Crucially for complexity questions, the alphabet (as it is referred to in [24], i.e. the set of propositions  $\text{Prop}$ ) that we consider is part of the input, it is not fixed beforehand.
- The bound  $B$  that we consider is written in unary instead of binary.

We discuss below the motivations behind our choices, and the implications they have on the complexity of the learning problems.

**Letters.** Using subsets of proposition as letters is not unusual, this is done for instance in the seminal book [4]. Furthermore, in this paper, we focus on comparing LTL, CTL and ATL learning. In that regard, we aim at having settings for all of these logics as close as possible. Hence, we use subsets of propositions as letters, since the states of Kripke structures are usually labeled with subsets of propositions (and the same goes for concurrent game structures). Note that this choice has significant impact on the complexity of the learning problem considered. Indeed, in our setting, letters are subsets of propositions. In combination with having the set of propositions as part of the input — which we discuss below — we show, in particular, that LTL learning with only operator  $\forall$  is NP-hard. On the other hand, in the setting of [24], the authors show that with only the operator  $\forall$ , the LTL learning problem can be decided in polynomial time (Proposition 7).

**Word length.** In this paper, we consider infinite words. For representation issues, we focus on ultimately periodic ones. As above, this is more closely related to CTL and ATL semantics. It has little influence on complexity questions, although some differences may arise. For instance, as stated in [24, Proposition 8], an LTL-formula of the shape  $\varphi := \mathbf{X}^k \varphi'$  for some  $k \in \mathbb{N}$  is always false when evaluated on words of size at most  $k - 1$ , which is irrelevant for us since all words have infinite size.

**Alphabet.** Let us now consider this more complicated issue: is it better to have the alphabet (or set of propositions) fixed a priori or to have it part of the input? As is mentioned in [24], it is much more usual to have the alphabet fixed a priori, as in the classical examples of automata learning ([17]). However, we believe that in a learning setting, it makes sense not to know a priori the propositions occurring in the model. That way, the set of propositions could be learned by looking at the models, positive or negative, that we need to separate<sup>1</sup>.

This choice is important in terms of complexity. Indeed, consider the very technical results of [24], in Section 7 and especially in Section 8, which deal with NP-hardness results (and hard-to-approximate results) with and without the next operator  $\mathbf{X}$ . These results are especially hard to prove because the set of propositions is fixed a priori (and is of size 3). On the other hand, our proofs of NP-hardness for LTL learning are significantly easier than, for instance, the proof of Theorem 9 from [24]. However, note that the distinction between having the set of propositions part of the input or not becomes less relevant when we restrict ourselves to formulas (for LTL, CTL and ATL learning) with bounded occurrence of binary operators, since in that case our NP-hardness proofs hold even for formula that do not use at all binary operators, and therefor use exactly one proposition.

**Bound.** Finally, there is the issue of the representation of the integer  $B$  that bounds the size of the formulas that we consider. In this paper, we consider the case where it is given in unary. Indeed, although we consider only decision problems where we only answer whether or not there exists a formula, it would also be interesting to explicitly synthesize formulas separating negative and positive instances. If the bound were written in binary, explicitly writing the formula could be exponential in the size of the input. Additionally, having the bound written in unary allows us to have completeness results, and not only hardness ones.

**Results from [24]** Let us now briefly discuss the NP-hardness results of [24]. Letting  $\text{Op}$  denote the operators allowed in LTL-formulas, they show that the following LTL learning problems are NP-hard:

- With alphabet part of the input:
  - When  $\text{Op} = \{\mathbf{F}, \vee\}$  (Theorem 2)
  - When  $\text{Op} = \{\mathbf{F}, \wedge\}$  (Theorem 8), and the learning problem is hard to approximate;
- With alphabet not part of the input:
  - When  $\{\mathbf{X}, \wedge, \vee\} \subseteq \text{Op} \subseteq \{\wedge, \vee, \mathbf{X}, \mathbf{F}, \mathbf{G}\}$  (Theorem 6, Proposition 11, Proposition 12);
  - When  $\{\mathbf{F}, \wedge\} \subseteq \text{Op} \subseteq \{\wedge, \vee, \mathbf{F}, \mathbf{G}, \neg\}$  (Theorem 9);
  - When  $\{\mathbf{G}, \vee\} \subseteq \text{Op} \subseteq \{\wedge, \vee, \mathbf{F}, \mathbf{G}, \neg\}$  (Theorem 10).

Note that, in a previous version of this paper, independently of [24], we have shown that the LTL learning problem is NP-hard for  $\{\vee\} \subseteq \text{Op} \subseteq \{\wedge, \vee, \Rightarrow, \Leftrightarrow, \mathbf{X}, \mathbf{F}, \mathbf{G}, \mathbf{U}, \mathbf{R}, \mathbf{W}, \mathbf{M}\}$ <sup>2</sup> (with the setting used in this paper). The reduction was established from the satisfiability problem SAT. This made the proof quite convoluted as, from a positive instance of the learning problem, we had to be able to extract to a satisfying valuation of the variables. On the other hand, in

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<sup>1</sup>Note that in our setting the set of propositions is given as an explicit part of the input. It could alternatively be given implicitly in the input as in the can read in the models.

<sup>2</sup>We present this way our result to mimic [24], however in this paper we distinguish between the logical and temporal operators that we consider.

[24], the authors often use the hitting set problem (one of Karp’s 21 NP-complete problem) for their reductions. We define it below.

**Definition 7** (Hitting set problem). *We denote by Hit the following decision problem:*

- *Input:*  $(l, C, k)$  where  $l \in \mathbb{N}_1$ ,  $C = C_1, \dots, C_n$  are non-empty subsets of  $[1, \dots, l]$  such that  $\cup_{1 \leq i \leq n} C_i = [1, \dots, l]$  and  $1 \leq k \leq l$ .
- *Output:* yes iff there is a subset  $H \subseteq [1, \dots, l]$  of size at most  $k$  such that, for all  $1 \leq i \leq n$ , we have  $H \cap C_i \neq \emptyset$ .

In the following, unless otherwise stated, when we use  $(l, C, k)$  as an instance of the hitting set problem,  $C$  refers to  $C = C_1, \dots, C_n$ , for some  $n \in \mathbb{N}_1$ .

**Theorem 8** ([21]). *The hitting set problem is NP-hard.*

**Observation 9.** *The above theorem holds even if  $k$  is given in unary. This comes from the fact that  $k \leq l$  and  $\cup_{1 \leq i \leq n} C_i = [1, \dots, l]$ .*

This problem is much better suited for establishing the NP-hardness of the learning problem decision that we consider. Indeed, it is sufficient to be able to exhibit a set of integers, not a valuation on variables. In particular, it makes it easier to handle various logical operators. In this paper, all the NP-hardness proofs that we exhibit in this paper but one are established from the hitting set problem<sup>3</sup>. Note that some of the NP-hardness proofs for the LTL learning case are very close to some proof from [24]. We discuss it in details below.

### 3.2 Our results

The goal of this subsection is to show the theorem below:

**Theorem 10.** *Consider some  $B^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$ , and  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and assume that  $B^l \cup B^t \neq \emptyset$ . Then, for all  $U^t \subseteq \text{Op}_{U_n}$ , the decision problem  $\text{LTL}_{\text{Learn}}(U^t, B^t, B^l, \infty)$  is NP-hard.*

Overall, there are fourteen different binary operators that we will handle, the ten logical operators  $\{\vee, \Rightarrow, \Leftarrow, \wedge, \neg\Rightarrow, \neg\Leftarrow, \neg\vee, \neg\wedge, \Leftrightarrow, \neg\Leftrightarrow\}$  and the four temporal operators  $\{\mathbf{U}, \mathbf{R}, \mathbf{W}, \mathbf{M}\}$ . We give below a bird’s eye view of how the proof of Theorem 10 is structured.

- We start with the operators  $\vee, \Rightarrow, \Leftarrow$ , i.e. we assume that  $B^l \cap \{\vee, \Rightarrow, \Leftarrow\} \neq \emptyset$ , and we show that for all  $B^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$  and  $U^t \subseteq \text{Op}_{U_n}$ , the decision problem  $\text{LTL}_{\text{Learn}}(U^t, B^t, B^l, \infty)$  is NP-hard. This is stated in Corollary 17. The reduction for this case is actually a straightforward adaptation of the proof of [24, Theorem 2] (that additionally makes use of the fact that our letters are subsets of propositions). In fact, the operators  $\neg\Rightarrow, \neg\Leftarrow, \neg\vee$  are handled at the same time. The reduction for these operators is obtained from the previous one by reversing the positive and negative sets of words.
- We then handle the operators  $\neg\vee, \neg\wedge$  (Corollary 23). The reduction used for the previous item cannot be used as is, because when the operator  $\neg\wedge$  (or the operator  $\neg\vee$ ) is used successively, the formula obtained is semantically equivalent to an alternation of conjunction and disjunction. An illustrating example is given in Example 18. To circumvent this difficulty, we define a reduction that is both slightly more subtle and an adaptation of the previous one.

<sup>3</sup>Except for the proof that CTL and ATL learning with any non-unary binary operator is NP-hard, which is proved via a reduction to the LTL case

- Before considering the last two logical operators  $\Leftrightarrow, \neg\Leftrightarrow$ , we handle the temporal operators  $\mathbf{W}, \mathbf{M}$  (Corollary 27). This is actually quite straightforward. Indeed, the two previous reductions only use size-1 infinite words (i.e. a subset of propositions is repeated indefinitely). On such words, the temporal operators  $\mathbf{W}, \mathbf{M}$  actually behave like  $\vee, \wedge$  respectively. Hence, we can use the reduction of the first item.
- We then handle the final two logical operators  $\Leftrightarrow, \neg\Leftrightarrow$  (Corollary 36). These operators behave quite differently from all other operators. In fact, in this case the reduction is not established from the hitting set problem, but from an NP-complete problem dealing with modulo-2 calculus (see Definition 32), although it still uses only size-1 infinite words. Contrary to the other reductions, here we explicitly assume that the only non-binary operators considered are  $\Leftrightarrow$  and  $\neg\Leftrightarrow$ .
- Finally, we handle the last two operators: the temporal operators  $\mathbf{U}$  and  $\mathbf{R}$  (Corollary 42). Contrary to the temporal operators  $\mathbf{W}$  and  $\mathbf{M}$ , on size-1 words,  $\mathbf{U}$  and  $\mathbf{R}$  are equivalent to unary binary operators. Hence, the reduction that we consider does not use only size-1 infinite words. It is once again established from the hitting set problem, however the construction is more involved than the reductions of the two first items.

Overall, note that we provide in this subsection a reduction for all possible binary operators to properly justify the statement: “if the occurrence of any binary operator is unbounded, the LTL learning decision problem is NP-hard”. However, binary operators are not all equivalently relevant, and it may be that some binary operator are not relevant at all when considered alone. This seems to be particularly the case for the operators  $\Leftrightarrow, \neg\Leftrightarrow$ .

### 3.2.1 Two useful lemmas

Before we handle all possible operators as described above, we first state and prove two lemmas that we will use extensively in this subsection.

First, we state a lemma that establishes a condition on the set of propositions occurring in an LTL-formula that distinguishes a pair of infinite words.

**Lemma 11.** *Consider a subset of propositions  $Y \subseteq \text{Prop}$  and two infinite words  $w_1, w_2 \in (2^{\text{Prop}})^\omega$ . Assume that, for all  $i \in \mathbb{N}_1$  and  $x \in \text{Prop} \setminus Y$ , we have  $x \in w_1[i]$  if and only if  $x \in w_2[i]$ . Then, if the formula  $\varphi$  distinguishes the words  $w_1$  and  $w_2$ , the set  $Y$  occurs in  $\varphi$ .*

*Proof.* Let us prove this property  $\mathcal{P}(\varphi)$  on LTL-formulas  $\varphi$  by induction. Consider such an LTL-formula  $\varphi$ :

- Assume that  $\varphi = x$  for some  $x \in \text{Prop}$ . Then, if  $\varphi$  distinguishes  $w_1$  and  $w_2$ , it must be that  $x \in Y$ , by assumption.
- Assume that  $\varphi = \neg\varphi'$  and  $\mathcal{P}(\varphi')$  holds. Then, if  $\varphi$  distinguishes  $w_1$  and  $w_1$ , so does  $\varphi'$ . Hence,  $\mathcal{P}(\varphi)$  follows.
- Assume that  $\varphi = \varphi_1 * \varphi_2$  for some  $*$   $\in \text{Op}_{\text{Bin}}^{\text{lg}}$  and that  $\mathcal{P}(\varphi_1)$  and  $\mathcal{P}(\varphi_2)$  hold. Then, if neither  $\varphi_1$  nor  $\varphi_2$  distinguish  $w_1$  and  $w_2$ , neither does  $\varphi$ . Hence, assuming that  $\varphi$  distinguishes  $w_1$  and  $w_2$ , then  $Y$  occurs in  $\varphi_1$  or  $\varphi_2$ , and it therefore also occurs in  $\varphi$ . Hence,  $\mathcal{P}(\varphi)$  holds.
- Assume that  $\varphi = \bullet\varphi'$  for some  $\bullet \in \text{Op}_{\text{Un}} \setminus \{\neg\}$  and  $\mathcal{P}(\varphi')$  holds. Assume that for all  $i \in \mathbb{N}_1$ ,  $\varphi'$  does not distinguish  $w_1[i : ]$  and  $w_2[i : ]$ . Then,  $\varphi'$  does not distinguish  $w_1[2 : ]$  and  $w_2[2 : ]$ , and  $\mathbf{X}\varphi'$  does not distinguish  $w_1$  and  $w_2$ . Furthermore, there is some  $i \in \mathbb{N}_1$

such that  $\varphi'$  accepts  $w_1[i : ]$  iff there is some  $i \in \mathbb{N}_1$  such that  $\varphi'$  accepts  $w_2[i : ]$ . That is,  $\mathbf{F} \varphi'$  does not distinguish  $w_1$  and  $w_2$ . This is similar for  $\mathbf{G} \varphi'$ . Hence, if  $\varphi$  distinguishes  $w_1$  and  $w_2$ ,  $Y$  occurs in  $\varphi'$ , and also in  $\varphi$ .

- The case of binary temporal operators is similar.

□

In addition, we show the straightforward relation between the number of propositions occurring in an LTL-formula and the size of that LTL-formula.

**Lemma 12.** *For all  $Y \subseteq \text{Prop}$ , we let  $\text{Occ}(Y), \text{NbSubF}(Y)$  denote the properties on LTL-formulas such that an LTL-formula  $\varphi$  satisfies:*

- $\text{Occ}(Y)$  if all variables in  $Y$  occur in  $\varphi$ , i.e.  $Y \subseteq \text{Prop}(\varphi)$ ;
- $\text{NbSubF}(Y)$  if there are at least  $2|Y| - 1$  different sub-formulas of  $\varphi$  where  $Y$  occurs.

*Then, an LTL-formula  $\varphi$  on  $\text{Prop}$  that satisfies  $\text{Occ}(Y)$  also satisfies  $\text{NbSubF}(Y)$ .*

*Proof.* Let us show this lemma by induction on LTL-formulas  $\varphi$ :

- Assume that  $\varphi = x$ . Then,  $\varphi$  satisfies  $\text{Occ}(Y)$  only for  $Y = \emptyset$  and  $Y = \{x\}$ , and it also satisfies  $\text{NbSubF}(\emptyset)$  and  $\text{NbSubF}(\{x\})$ ;
- for all  $\bullet \in \text{Op}_{\text{Un}}$ , assume that  $\varphi = \bullet\varphi'$ . Consider any  $Y \subseteq \text{Prop}$ . If  $\varphi$  satisfies  $\text{Occ}(Y)$ , then so does  $\varphi'$ . Hence,  $\varphi'$  satisfies  $\text{NbSubF}(Y)$ , and therefore so does  $\varphi$ ;
- for all  $\bullet \in \text{Op}_{\text{Bin}}^{\text{lg}} \cup \text{Op}_{\text{Bin}}^{\text{tp}}$ , assume that  $\varphi = \varphi_1 \bullet \varphi_2$ . Consider any  $Y \subseteq \text{Prop}$  and assume that  $\varphi$  satisfies  $\text{Occ}(Y)$ . Let  $Y_1 := \text{Prop}(\varphi_1) \cap Y \subseteq Y$  denote the set of different propositions in  $Y$  occurring in  $\varphi_1$  (they may also occur in  $\varphi_2$ ). Let also  $Y_2' := (\text{Prop}(\varphi_2) \setminus \text{Prop}(\varphi_1)) \cap Y \subseteq Y$  denote the set of different propositions in  $Y$  occurring in  $\varphi_2$  and not in  $\varphi_1$ . We have  $Y = Y_1 \cup Y_2'$ . Furthermore:
  - by our induction hypothesis on  $\varphi_1$ , there are at least  $k_1 := 2|Y_1| - 1$  sub-formulas in  $\text{SubF}(\varphi_1) \subseteq \text{SubF}(\varphi)$  where  $Y_1$  occurs;
  - by our induction hypothesis on  $\varphi_2$ , there are also at least  $k_2' := 2|Y_2'| - 1$  sub-formulas in  $\text{SubF}(\varphi_2) \subseteq \text{SubF}(\varphi)$  where  $Y_2'$  occurs. By definition of  $Y_2'$ , it follows that all these sub-formulas are not sub-formulas of  $\varphi_1$ ;
  - Finally, the sub-formula  $\varphi$  itself is a sub-formula of  $\varphi$  that is neither a sub-formula of  $\varphi_1$  nor of  $\varphi_2$  and where  $Y$  occurs.

Therefore, there are at least  $k_1 + k_2' + 1 = 2|Y_1| - 1 + 2|Y_2'| - 1 + 1 \geq 2|Y| - 1$  different sub-formulas of  $\varphi$  where  $Y$  occurs. That is,  $\varphi$  satisfies  $\text{NbSubF}(Y)$ .

□

### 3.2.2 Proof of Theorem 10: when $\mathbf{B}^! \cap \{\vee, \Rightarrow, \Leftarrow, \wedge, \neg\Rightarrow, \neg\Leftarrow\} \neq \emptyset$

We present a first reduction that we will consider for the operators  $\vee, \Rightarrow, \Leftarrow$ , along with the dual reduction for the operators  $\wedge, \neg\Rightarrow, \neg\Leftarrow$ . This is obtained via a slight modification of the reduction presented in [24] to establish Theorem 2. We make use of the fact that the letters we consider are subsets of propositions instead of being a single proposition.

**Definition 13.** *Consider an instance  $(l, C, k)$  of the hitting set problem  $\text{Hit}$ . We define:*

- $\text{Prop} := \{a_j, b_j \mid 1 \leq j \leq l\}$  to be the set of propositions;
- $\text{Set} := \{\alpha_i \mid 1 \leq i \leq n\}$  where for all  $1 \leq i \leq n$ : we let  $\alpha_i := \{c_i^1, \dots, c_i^l\}^\omega \in (2^{\text{Prop}})$  with, for all  $1 \leq j \leq l$ :

$$c_i^j := \begin{cases} a_j & \text{if } j \in C_i \\ b_j & \text{if } j \notin C_i \end{cases}$$

- $\text{EmptySet} := \{\beta\}$  with  $\beta := \{b_1, \dots, b_l\}^\omega \in (2^{\text{Prop}})$ ;
- $B = 2k - 1$ .

Then, we define the inputs  $\text{In}_{(l,C,k)}^\vee := (\text{Prop}, \text{Set}, \text{EmptySet}, B)$  and  $\text{In}_{(l,C,k)}^\wedge := (\text{Prop}, \text{EmptySet}, \text{Set}, B)$ .

The positive and negative words that we have defined above satisfy the observation below.

**Observation 14.** For all  $w \in \text{EmptySet} \cup \text{Set}$ , we have:

$$\forall 1 \leq j \leq l, a_j \models w \Leftrightarrow \neg b_j \models w$$

Let us describe on an example below what this reduction amounts to on a specific instance of the hitting set problem.

**Example 15.** Assume that  $l = 4$ ,  $C = C_1, C_2$  with  $C_1 := \{1, 3\}$  and  $C_2 := \{1, 2, 4\}$  and  $k = 1$ . Then, we have:  $\text{Prop} = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4\}$ ,  $\alpha_1 = (\{a_1, b_2, a_3, b_4\})^\omega$ ,  $\alpha_2 = (\{a_1, a_2, b_3, a_4\})^\omega$  and  $\beta = (\{b_1, b_2, b_3, b_4\})^\omega$ . Finally,  $B = 1$ .

The above definition satisfies the lemma below.

**Lemma 16.** Consider an instance  $(l, C, k)$  is a positive instance of the hitting set problem  $\text{Hit}$  and sets of operators  $\text{U}^t \subseteq \text{Op}_{\text{Un}}$ ,  $\text{B}^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$ , and  $\text{B}^l \subseteq \text{Op}_{\text{Bin}}$ . If  $\text{In}_{(l,C,k)}^\vee$  or  $\text{In}_{(l,C,k)}^\wedge$  is a positive instance of the decision problem  $\text{LTL}_{\text{Learn}}(\text{U}^t, \text{B}^t, \text{B}^l, \infty)$ , then  $(l, C, k)$  is a positive instance of the hitting set problem  $\text{Hit}$ .

On the other hand, if  $(l, C, k)$  is a positive instance of the hitting set problem  $\text{Hit}$  and  $\{\vee, \Rightarrow, \Leftarrow\} \cap \text{B}^l \neq \emptyset$  (resp.  $\{\wedge, \neg \Rightarrow, \neg \Leftarrow\} \cap \text{B}^l \neq \emptyset$ ), then  $\text{In}_{(l,C,k)}^\vee$  (resp.  $\text{In}_{(l,C,k)}^\wedge$ ) is a positive instance of the decision problem  $\text{LTL}_{\text{Learn}}(\text{U}^t, \text{B}^t, \text{B}^l, \infty)$ .

*Proof.* Assume that  $\text{In}_{(l,C,k)}^\vee$  or  $\text{In}_{(l,C,k)}^\wedge$  is a positive instance of  $\text{LTL}_{\text{Learn}}(\text{U}^t, \text{B}^t, \text{B}^l, \infty)$ . Consider an LTL-formula  $\varphi$  of size at most  $B = 2k - 1$  that distinguishes the sets of infinite words  $\text{Set}$  and  $\text{EmptySet}$ . Let  $H = \{i \in [1, \dots, l] \mid \{a_i, b_i\} \cap \text{Prop}(\varphi) \neq \emptyset\}$  denote the set of integers  $i$  for which at least one of the corresponding variables  $a_i$  or  $b_i$  occurs in  $\varphi$ . By Lemma 12, it must be that  $|H| \leq k$  since  $|\varphi| \leq 2k - 1$ . Let us show that  $H$  is a hitting set. Let  $1 \leq i \leq n$ . The formula  $\varphi$  distinguishes the infinite words  $\alpha_i$  and  $\beta$ . Furthermore, for all  $j \in [1, \dots, l] \setminus C_i$ , we have  $b_j \in \alpha_i[1]$  and  $b_j \in \beta[1]$  (also, recall Observation 14). Hence, by Lemma 11, it must be that  $C_i \cap H \neq \emptyset$ . Since this holds for all  $1 \leq i \leq n$ , we obtain that  $H$  is indeed a hitting set. Note that the arguments that we have given here hold regardless of the operators used in the formula  $\varphi$ . Hence,  $(l, C, k)$  is a positive instance of the hitting set problem  $\text{Hit}$ .

Assume now that  $(l, C, k)$  is a positive instance of the hitting set problem  $\text{Hit}$ . Consider a hitting set  $H \subseteq [1, \dots, l]$  of size at most  $k$ . We denote  $H := \{j_1, \dots, j_r\}$  with  $|H| = r \leq k$ . We define LTL-formulas indexed by the operator that we consider.

- $\varphi_\vee := a_{j_1} \vee a_{j_2} \vee \dots \vee a_{j_r}$

- $\varphi_{\Rightarrow} := b_{j_1} \Rightarrow (b_{j_2} \Rightarrow (\dots \Rightarrow a_{j_r}))$
- $\varphi_{\Leftarrow} := ((a_{j_1} \Leftarrow b_{j_2}) \Leftarrow \dots) \Leftarrow b_{j_r}$

Recall that for all  $x_1, x_2 \in \mathbb{B}$ , we have  $x_1 \Rightarrow x_2 = \neg x_1 \vee x_2$  and  $x_1 \Leftarrow x_2 = x_1 \vee \neg x_2$ , hence, by Observation 14, for all  $w \in \text{EmptySet} \cup \text{Set}$ , we have  $w \models \varphi_{\vee}$  iff  $w \models \varphi_{\Rightarrow}$  iff  $w \models \varphi_{\Leftarrow}$ .

We also define the LTL-formulas below.

- $\varphi_{\wedge} := b_{j_1} \wedge b_{j_2} \wedge \dots \wedge b_{j_r}$
- $\varphi_{\neg\Leftarrow} := a_{j_1} \neg\Leftarrow (a_{j_2} \neg\Leftarrow (\dots \neg\Leftarrow b_{j_r}))$
- $\varphi_{\neg\Rightarrow} := ((b_{j_1} \neg\Rightarrow a_{j_2}) \neg\Rightarrow \dots) \neg\Rightarrow a_{j_r}$

Recall that for all  $x_1, x_2 \in \mathbb{B}$ , we have  $x_1 \neg\Rightarrow x_2 = x_1 \wedge \neg x_2$  and  $x_1 \neg\Leftarrow x_2 = \neg x_1 \wedge x_2$ . Hence, by Observation 14, for all  $w \in \text{EmptySet} \cup \text{Set}$ , we have  $w \models \varphi_{\wedge}$  iff  $w \models \varphi_{\neg\Rightarrow}$  iff  $w \models \varphi_{\neg\Leftarrow}$  and  $w \models \varphi_{\wedge}$  iff  $w \not\models \varphi_{\vee}$ .

Clearly, we have  $|\varphi_{\vee}| = |\varphi_{\Rightarrow}| = |\varphi_{\Leftarrow}| = |\varphi_{\wedge}| = |\varphi_{\neg\Rightarrow}| = |\varphi_{\neg\Leftarrow}| = 2r - 1 \leq B$ .

Furthermore, consider any  $1 \leq i \leq n$ . Let  $j \in H \cap C_i \neq \emptyset$ . We have  $\alpha_i \models a_j$ , hence  $\alpha_i \models \varphi_{\vee}$ . This holds for all  $1 \leq i \leq n$ . Furthermore, we also have  $\beta \not\models \varphi_{\vee}$ . Therefore,  $\varphi_{\vee}$  accepts **Set** and rejects **EmptySet**. It is also the case for  $\varphi_{\Rightarrow}$  and  $\varphi_{\Leftarrow}$ . It is the opposite for the formulas  $\varphi_{\wedge}, \varphi_{\neg\Rightarrow}, \varphi_{\neg\Leftarrow}$  (i.e. they accept **EmptySet** and reject **Set**). Hence, if  $\{\vee, \Rightarrow, \Leftarrow\} \cap \mathbf{B}^l \neq \emptyset$  then  $\text{In}_{(l,C,k)}^{\vee}$  is a positive instance of  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  and if  $\{\wedge, \neg\Rightarrow, \neg\Leftarrow\} \cap \mathbf{B}^l \neq \emptyset$ , then  $\text{In}_{(l,C,k)}^{\wedge}$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$ .  $\square$

We obtain the corollary below.

**Corollary 17.** *Consider a set  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  of binary logical operators and assume that  $\mathbf{B}^t \cap \{\vee, \Rightarrow, \Leftarrow, \wedge, \neg\Rightarrow, \neg\Leftarrow\} \neq \emptyset$ . For all  $\mathbf{U}^t \subseteq \text{Op}_{\text{Un}}$  and  $\mathbf{B}^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$ , the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem is NP-hard.*

*Proof.* This is a direct consequence of Lemmas 16 and the fact that the instances  $\text{In}_{(l,C,k)}^{\neg\vee}$  and  $\text{In}_{(l,C,k)}^{\neg\wedge}$  can be computed in logarithmic space from  $(l, C, k)$ .  $\square$

### 3.2.3 Proof of Theorem 10: when $\mathbf{B}^l \cap \{\neg\vee, \neg\wedge\} \neq \emptyset$ .

The case of the operators  $\neg\vee$  and  $\neg\wedge$  is slightly different. Indeed, contrary to the above operators, when successively using one of these operators, we obtain (semantically) an alternation of conjunctions and disjunctions. We describe it on an example below.

**Example 18.** *Consider six variables  $x_1, x_2, x_3, x_4, x_5, x_6$ . Assume that we want to use them in a single LTL-formula. If we can use the  $\vee$  operator, we can consider the formula  $x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5 \vee x_6$ . If we can use the  $\Rightarrow$  operator, we can consider the formula  $x_1 \Rightarrow (x_2 \Rightarrow (x_3 \Rightarrow (x_4 \Rightarrow (x_5 \Rightarrow x_6))))$ . Note that, up to some negation on the variables, this amounts semantically to only using the  $\vee$  operator. However, assume now that we can only use the  $\neg\wedge$  operator. For instance, consider:*

$$\varphi := x_1 \neg\wedge (x_2 \neg\wedge (x_3 \neg\wedge (x_4 \neg\wedge (x_5 \neg\wedge x_6))))$$

*It is semantically equivalent to:*

$$\varphi \equiv \neg x_1 \vee (x_2 \wedge (\neg x_3 \vee (x_4 \wedge (\neg x_5 \vee \neg x_6))))$$

*Here, we have both operators  $\vee$  and  $\wedge$ .*

To circumvent this difficulty, we are going to change the reduction by adding propositions that will always hold on the words of interest. We can then place these propositions where  $x_2$  and  $x_4$  were in the above formula. That way, on the infinite words where these propositions hold, we obtain a disjunction, as in the formula above.

We start with the reduction for the  $\neg \wedge$  operator.

**Definition 19.** Consider an instance  $(l, C, k)$  of the hitting set problem **Hit**. If  $k \geq l$ ,  $(l, C, k)$  is obviously a positive instance of the hitting set problem **Hit**, and we define  $\text{In}_{(l, C, k)}^{\neg \wedge}$  to be an arbitrary positive instance of the LTL learning decision problem. Otherwise, we define:

- $\text{Prop} := \{a_i, b_i \mid 1 \leq i \leq l\} \cup \{x_i \mid 1 \leq i \leq k\}$  to be the set of propositions;
- $\mathcal{P} := \{\alpha_1, \dots, \alpha_n, \beta\}$  where for all  $1 \leq i \leq n$ : we let  $\alpha_i := \{x_1, \dots, x_k, c_i^1, \dots, c_i^l\}^\omega \in (2^{\text{Prop}})$  with, for all  $1 \leq j \leq l$ :

$$c_i^j := \begin{cases} a_j & \text{if } j \in C_i \\ b_j & \text{if } j \notin C_i \end{cases}$$

and  $\beta := \{x_1, \dots, x_{k-1}, b_1, \dots, b_l\}^\omega$ ;

- $\mathcal{N} := \{\alpha, \beta_1, \dots, \beta_k\}$  with  $\alpha := \{x_1, \dots, x_k, b_1, \dots, b_l\}^\omega \in (2^{\text{Prop}})$  and, for all  $1 \leq i \leq k-1$ , we have  $\beta_i := (\{x_1, \dots, x_{k-1}, b_1, \dots, b_l\} \setminus \{x_i\})^\omega$  and  $\beta_k := \{x_1, \dots, x_k, b_1, \dots, b_l\}^\omega$ ;
- $B = 4k - 1$ .

Then, we define the input  $\text{In}_{(l, C, k)}^{\neg \wedge} := (\text{Prop}, \mathcal{P}, \mathcal{N}, B)$  of the  $\text{LTL}_{\text{Learn}}$  decision problem.

Similarly to the previous reduction, we have the following lemma.

**Lemma 20.** Consider a set  $\mathbf{B}^l$  of binary logical operators and assume that  $\neg \wedge \in \mathbf{B}^l$ . Then, for all  $\mathbf{U}^t \subseteq \text{Op}_{\text{Un}}^t$  and  $\mathbf{B}^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$ ,  $(l, C, k)$  is a positive instance of the hitting set problem **Hit** if and only if  $\text{In}_{(l, C, k)}^{\neg \wedge}$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem.

This proof of this lemma is quite close to the proof of Lemma 16.

*Proof.* If  $k \geq l$ , the equivalence is straightforward. We assume in the following that  $k \leq l$ .

Assume that  $(l, C, k)$  is a positive instance of the hitting set problem **Hit**. Consider a hitting set  $H \subseteq [1, \dots, l]$  of size at most  $k$ . Consider any set  $H' \subseteq [1, \dots, l]$  of size exactly  $k$  such that  $H \subseteq H'$ . We let  $H' := \{j_1, \dots, j_k\}$ . We let:

- $\varphi_{\neg \wedge} := b_{j_1} \neg \wedge (x_1 \neg \wedge (b_{j_2} \neg \wedge (\dots x_{k-1} \neg \wedge (b_{j_k} \neg \wedge x_k))))$
- $\varphi_{\neg \wedge}^{\text{expl}} := \neg b_{j_1} \vee (x_1 \wedge (\neg b_{j_2} \vee (\dots x_{k-1} \wedge (\neg b_{j_k} \vee \neg x_k))))$

The formula  $\varphi_{\neg \wedge}^{\text{expl}}$  is written to make more explicit what  $\varphi_{\neg \wedge}$  is equal to. Indeed, recall that for all  $x_1, x_2 \in \mathbb{B}$ , we have  $x_1 \neg \wedge x_2 = \neg x_1 \vee \neg x_2$ , hence, for all  $w \in \mathcal{P} \cup \mathcal{N}$ , we have  $w \models \varphi_{\neg \wedge}$  iff  $w \models \varphi_{\neg \wedge}^{\text{expl}}$ . Furthermore, we have  $|\varphi_{\neg \wedge}| = 4k - 3 = B$ .

Let  $\gamma \in \{\alpha, \alpha_1, \dots, \alpha_n\}$ . We have  $\gamma \models x_1 \wedge \dots \wedge x_k$ . Hence,  $\gamma \models \varphi_{\neg \wedge}$  if and only if  $\gamma \models \neg b_{j_1} \vee \dots \vee \neg b_{j_k}$ . Therefore,  $\alpha \not\models \varphi_{\neg \wedge}$ . However, for any  $1 \leq i \leq n$ , there is  $j \in H \cap C_i \neq \emptyset$  such that  $\alpha_i \models \neg b_j$ , and thus  $\alpha_i \models \varphi_{\neg \wedge}$ . In addition, consider any  $\delta \in \{\beta, \beta_1, \dots, \beta_k\}$ . For all  $j \in [1, \dots, l]$ , we have  $\delta \models b_j$ . Hence,  $\delta \models \varphi_{\neg \wedge}$  if and only if  $\delta \models x_1 \wedge \dots \wedge x_{k-1} \wedge \neg x_k$ . Hence,  $\beta \models \varphi_{\neg \wedge}$  whereas, for all  $1 \leq i \leq k$ , we have  $\beta_i \not\models \varphi_{\neg \wedge}$ . Overall, the formula  $\varphi_{\neg \wedge}$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . Hence, the decision problem  $\text{In}_{(l, C, k)}^{\neg \wedge}$  is a positive instance of  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$ .



Assume now the decision problem  $\text{In}_{(l,C,k)}^{\neg\wedge}$  is a positive instance of  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$ . Consider an LTL-formula  $\varphi$  of size at most  $B = 4k - 1$  that distinguishes the sets of infinite words  $\mathcal{P}$  and  $\mathcal{N}$ . Let  $H = \{i \in [1, \dots, l] \mid \{a_i, b_i\} \cap \text{Prop}(\varphi) \neq \emptyset\}$  denote the set of integers  $i$  for which at least one of the corresponding variables  $a_i$  or  $b_i$  occurs in  $\varphi$ . Let us show that this set is of size at most  $k$  and intersects all sets  $C_i$ .

By Lemma 12, we have  $\text{Prop}(\varphi) \leq 2k$ . Furthermore, by Lemma 11, since for all  $1 \leq i \leq k$ , the formula  $\varphi$  distinguishes the words  $\beta$  and  $\beta_i$ , it follows that  $x_i \in \text{Prop}(\varphi)$ . Hence,  $|H| \leq 2k - k = k$ .

Furthermore, let  $1 \leq i \leq n$ . The formula  $\varphi$  distinguishes the infinite words  $\alpha_i$  and  $\alpha$ . In addition, for all  $j \in [1, \dots, l] \setminus C_i$ , we have  $b_j \in \alpha_i[1] \cap \alpha[1]$ ,  $a_j \notin \alpha_i[1] \cup \alpha[1]$  and  $\alpha_i[1] \cap \{x_1, \dots, x_k\} = \alpha[1] \cap \{x_1, \dots, x_k\}$ . Hence, by Lemma 11 and by Definition of  $\alpha_i$ , it must be that  $C_i \cap H \neq \emptyset$ . Since this holds for all  $1 \leq i \leq n$ , we obtain that  $H$  is indeed a hitting set. Hence,  $(l, C, k)$  is a positive instance of the hitting set problem Hit.  $\square$

Contrary to the reductions we defined in Definition 13, the reduction for the operator  $\neg\vee$  is not obtained from the reduction for  $\neg\wedge$  by reversing the positive and negative sets of words, though it is quite similar. We give it below.

**Definition 21.** Consider an instance  $(l, C, k)$  of the hitting set problem Hit. If  $k \geq l$ ,  $(l, C, k)$  is obviously a positive instance of the hitting set problem Hit, and we define  $\text{In}_{(l,C,k)}^{\neg\vee}$  to be an arbitrary positive instance of the LTL learning decision problem. Otherwise, we define:

- $\text{Prop} := \{a_i, b_i \mid 1 \leq j \leq l\} \cup \{x_i \mid 1 \leq i \leq k\}$  to be the set of propositions;
- $\mathcal{P} := \{\alpha, \beta_1, \dots, \beta_k\}$  with  $\alpha := \{b_1, \dots, b_l\}^\omega \in (2^{\text{Prop}})$  and, for all  $1 \leq i \leq k - 1$ , we have  $\beta_i := \{b_1, \dots, b_l, x_i, x_k\}^\omega$  and  $\beta_k = \{b_1, \dots, b_l\}^\omega$ ;
- $\mathcal{N} := \{\alpha_1, \dots, \alpha_n, \beta\}$  where for all  $1 \leq i \leq n$ : we let  $\alpha_i := \{c_i^1, \dots, c_i^l\}^\omega \in (2^{\text{Prop}})$  with, for all  $1 \leq j \leq l$ :

$$c_i^j := \begin{cases} a_j & \text{if } j \in C_i \\ b_j & \text{if } j \notin C_i \end{cases}$$

and  $\beta := \{b_1, \dots, b_l, x_k\}^\omega$ ;

- $B = 4k - 1$ .

Then, we define the inputs  $\text{In}_{(l,C,k)}^{\neg\vee} := (\text{Prop}, \mathcal{P}, \mathcal{N}, B)$  of the  $\text{LTL}_{\text{Learn}}$  decision problem.

Similarly to the previous reduction, we have the following lemma.

**Lemma 22.** Consider a set  $\mathbf{B}^l$  of binary logical operators and assume that  $\neg\vee \in \mathbf{B}^l$ . Then, for all  $\mathbf{U}^t \subseteq \text{Op}_{\text{Un}}$  and  $\mathbf{B}^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$ ,  $(l, C, k)$  is a positive instance of the hitting set problem Hit if and only if  $\text{In}_{(l,C,k)}^{\neg\vee}$  is a positive instance of the the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem.

The proof of this lemma is very close to the proof of Lemma 20. Hence, we only give the formula using only the  $\neg\vee$  operator built from a hitting set that we consider.

*Proof sketch.* If  $k \geq l$ , the equivalence is straightforward. Let us now assume that  $k \leq l$ .

Assume that  $(l, C, k)$  is a positive instance of the hitting set problem Hit. Consider a hitting set  $H \subseteq [1, \dots, l]$  of size at most  $k$ . Consider any set  $H' \subseteq [1, \dots, l]$  of size exactly  $k$  such that  $H \subseteq H'$ . We let  $H' := \{j_1, \dots, j_k\}$  and:

- $\varphi_{\neg\vee} := a_{j_1} \neg\vee (x_1 \neg\vee (a_{j_2} \neg\vee (\dots x_{k-1} \neg\vee (a_{j_k} \neg\vee x_k))))$

- $\varphi_{\neg\vee}^{\text{expl}} := \neg a_{j_1} \wedge (x_1 \vee (\neg a_{j_2} \wedge (\dots x_{k-1} \vee (\neg a_{j_k} \wedge \neg x_k))))$

The formula  $\varphi_{\neg\vee}^{\text{expl}}$  is written to make more explicit what  $\varphi_{\neg\vee}$  is equal to. Indeed, recall that for all  $x_1, x_2 \in \mathbb{B}$ , we have  $x_1 \neg\vee x_2 = \neg x_1 \wedge \neg x_2$ , hence, for all  $w \in \mathcal{P} \cup \mathcal{N}$ , we have  $w \models \varphi_{\neg\vee}$  iff  $w \models \varphi_{\neg\vee}^{\text{expl}}$ . One can then check that the formula  $\varphi_{\neg\vee}$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ .  $\square$

We deduce the corollary below.

**Corollary 23.** *Consider a set  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  of binary logical operators and assume that  $\mathbf{B}^t \cap \{\neg\wedge, \neg\vee\} \neq \emptyset$ . For all  $\mathbf{U}^t \subseteq \text{Op}_{\text{Un}}$  and  $\mathbf{B}^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$ , the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem is NP-hard.*

*Proof.* This is a direct consequence of Lemmas 20 and 22 and the fact that the instances  $\text{In}_{(l,C,k)}^{\neg\vee}$  and  $\text{In}_{(l,C,k)}^{\neg\wedge}$  can be computed in logarithmic space from  $(l, C, k)$ .  $\square$

### 3.2.4 Proof of Theorem 10: with the temporal operators **W** and **M**

The last two logical binary operators  $\Leftrightarrow$  and  $\neg\Leftrightarrow$  will be handled by considering a reduction that is completely different from what we have presented so far. However, to prove the correctness of that reduction we first show a way to handle (i.e. to remove them from the formulas that we consider) the temporal operators. This can be done because, in all the reductions that we have presented so far, and with the reduction for the operators  $\Leftrightarrow$  and  $\neg\Leftrightarrow$  that will follow, all the infinite words that we consider are size-1 infinite words. We define a notion of equivalence on LTL-formulas over these infinite words.

**Definition 24.** *Consider a set of propositions  $\text{Prop}$  and two LTL-formulas  $\varphi_1$  and  $\varphi_2$ . We write  $\varphi_1 \equiv_1 \varphi_2$  when, for all size-1 infinite words  $w \in (2^{\text{Prop}})^\omega$ , we have  $w \models \varphi_1$  if and only if  $w \models \varphi_2$ .*

Then, we have the following lemma.

**Lemma 25.** *Consider three LTL-formulas  $\varphi, \varphi_1, \varphi_2 \in \text{LTL}$ . We have:*

1. for all  $\bullet \in \{\mathbf{X}, \mathbf{F}, \mathbf{G}\}$ , we have  $\varphi \equiv_1 \bullet\varphi$ ;
2. for all  $\bullet \in \{\mathbf{U}, \mathbf{R}\}$ , we have  $\varphi_2 \equiv_1 \varphi_1 \bullet\varphi_2$ ;
3.  $\varphi_1 \vee \varphi_2 \equiv_1 \varphi_1 \mathbf{W} \varphi_2$ ;
4.  $\varphi_1 \wedge \varphi_2 \equiv_1 \varphi_1 \mathbf{M} \varphi_2$ .

*Proof.* Consider any  $\alpha \subseteq \text{Prop}$ . Note that for all  $i \in \mathbb{N}_1$ , we have  $\alpha^\omega[i:] = \alpha^\omega$ . Therefore:

1.
  - $\alpha^\omega \models \varphi$  iff  $\alpha^\omega[2:] \models \varphi$  iff  $\alpha^\omega \models \mathbf{X} \varphi$ ;
  - $\alpha^\omega \models \varphi$  iff  $\exists i \in \mathbb{N}_1, \alpha^\omega[i:] \models \varphi$  iff  $\alpha^\omega \models \mathbf{F} \varphi$ ;
  - $\alpha^\omega \models \varphi$  iff  $\forall i \in \mathbb{N}_1, \alpha^\omega[i:] \models \varphi$  iff  $\alpha^\omega \models \mathbf{G} \varphi$ ;
2. If  $\alpha^\omega \models \varphi_2$ , then  $\alpha^\omega \models \varphi_1 \mathbf{U} \varphi_2$ . Furthermore, if  $\alpha^\omega \models \varphi_1 \mathbf{U} \varphi_2$ , then there is some  $i \in \mathbb{N}_1$  such that  $\alpha^\omega[i:] \models \varphi_2$ . Hence,  $\alpha^\omega \models \varphi_2$ . In fact,  $\alpha^\omega \models \varphi_1 \mathbf{U} \varphi_2$  iff  $\alpha^\omega \models \varphi_2$ .

In addition, we have:

$$\alpha^\omega \models \varphi_1 \mathbf{R} \varphi_2 \text{ iff } \alpha^\omega \not\models \neg\varphi_1 \mathbf{U} \neg\varphi_2 \text{ iff } \alpha^\omega \not\models \neg\varphi_2 \text{ iff } \alpha^\omega \models \varphi_2$$

3.  $\alpha^\omega \models \varphi_1 \mathbf{W} \varphi_2$  iff  $\alpha^\omega \models (\varphi_1 \mathbf{U} \varphi_2) \vee \mathbf{G} \varphi_1$  iff  $\alpha^\omega \models \varphi_2 \vee \varphi_1$ ;

4.  $\alpha^\omega \models \varphi_1 \mathbf{M} \varphi_2$  iff  $\alpha^\omega \models (\varphi_1 \mathbf{R} \varphi_2) \wedge \mathbf{F} \varphi_1$  iff  $\alpha^\omega \models \varphi_2 \wedge \varphi_2$ .

□

In particular, this lemma tells us that, on size-1 infinite words, the  $\mathbf{W}$  temporal operator behaves like the  $\vee$  logical operator and the  $\mathbf{M}$  temporal operator behaves like the  $\wedge$  logical operator. Hence, we can reuse the reduction from Definition 13 to establish that the LTL learning problem is NP-hard with at least one of the two temporal operators  $\mathbf{W}$  or  $\mathbf{M}$ , as stated in the lemma below.

**Lemma 26.** *Consider a set  $\mathbf{B}^t$  of binary temporal operators and assume that  $\mathbf{W} \in \mathbf{B}^t$  (resp.  $\mathbf{M} \in \mathbf{B}^t$ ). Then, for all  $\mathbf{U}^t \subseteq \text{Op}_{\text{Un}}$  and  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$ ,  $(l, C, k)$  is a positive instance of the hitting set problem Hit if and only if  $\text{In}_{(l, C, k)}^\vee$  is a positive instance of the the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem (resp.  $\text{In}_{(l, C, k)}^\wedge$  is a positive instance of the the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$ ).*

*Proof.* The bottom to top implication is already given by Lemma 16. Furthermore, let us assume that  $(l, C, k)$  is a positive instance of the hitting set problem Hit. Then, from a hitting set  $H \subseteq [1, \dots, l]$  of size at most  $k$  with  $H := \{j_1, \dots, j_r\}$ , we consider the LTL-formulas:  $\varphi_{\mathbf{W}} := a_{j_1} \mathbf{W}(a_{j_2} \mathbf{W}(\dots \mathbf{W} a_{j_r}))$  and  $\varphi_{\mathbf{M}} := b_{j_1} \mathbf{M}(b_{j_2} \mathbf{M}(\dots \mathbf{M} b_{j_r}))$ . We have  $|\varphi_{\mathbf{W}}| = |\varphi_{\mathbf{M}}| = 2r - 1 \leq 2k - 1$ . In addition, by Lemma 25,  $\varphi_{\mathbf{W}} \equiv_1 a_{j_1} \vee \dots \vee a_{j_r} = \varphi_\vee$  and  $\varphi_{\mathbf{M}} := b_{j_1} \wedge \dots \wedge b_{j_r} = \varphi_\wedge$ , with  $\varphi_\vee$  and  $\varphi_\wedge$  from the proof of Lemma 16, where we have shown that  $\varphi_\vee$  accepts Set and rejects EmptySet and  $\varphi_\wedge$  accepts EmptySet and rejects Set. The lemma follows. □

We deduce the corollary below.

**Corollary 27.** *Consider a set  $\mathbf{B}^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$  of binary temporal operators with  $\mathbf{B}^t \cap \{\mathbf{W}, \mathbf{M}\} \neq \emptyset$ . For all  $\mathbf{U}^t \subseteq \text{Op}_{\text{Un}}$  and  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$ , the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem is NP-hard.*

*Proof.* This is a direct consequence of Lemma 26 and the fact that the instances  $\text{In}_{(l, C, k)}^\vee$  and  $\text{In}_{(l, C, k)}^\wedge$  can be computed in logarithmic space from  $(l, C, k)$ . □

### 3.2.5 Proof of Theorem 10: when $\mathbf{B}^l \cap \{\Leftrightarrow, \neg\Leftrightarrow\} \neq \emptyset$ .

We now consider the case of the operators  $\Leftrightarrow$  and  $\neg\Leftrightarrow$ . To handle this case, we are going to restrict ourselves to  $\Leftrightarrow$ -formula, defined below, i.e. LTL-formulas that only use operators  $\{\neg, \Leftrightarrow, \neg\Leftrightarrow\}$ .

**Definition 28.** *We say that an LTL-formula is a  $\Leftrightarrow$ -formula if it belongs to the fragment  $\text{LTL}(\{\neg\}, \{\Leftrightarrow, \neg\Leftrightarrow\}, \emptyset)$ .*

We exhibit a function that removes all the operators that we do not want to consider here.

**Lemma 29.** *There exists a function  $\text{tr} : \text{LTL}(\text{Op}_{\text{Un}}, \{\Leftrightarrow, \neg\Leftrightarrow\}, \{\mathbf{U}, \mathbf{R}\}) \rightarrow \text{LTL}(\{\neg\}, \{\Leftrightarrow, \neg\Leftrightarrow\}, \emptyset)$  such that, for all LTL-formulas  $\varphi \in \text{LTL}$ :*

1.  $\text{SubF}(\text{tr}(\varphi)) \subseteq \text{tr}[\text{SubF}(\varphi)] := \{\text{tr}(\varphi') \mid \varphi' \in \text{SubF}(\varphi)\}$ , hence  $|\text{tr}(\varphi)| \leq |\varphi|$ ;
2.  $\varphi \equiv_1 \text{tr}(\varphi)$ .

*Proof.* We define  $\text{tr} : \text{LTL}(\text{Op}_{\text{Un}}, \{\Leftrightarrow, \neg\Leftrightarrow\}, \{\mathbf{U}, \mathbf{R}\}) \rightarrow \text{LTL}(\{\neg\}, \{\Leftrightarrow, \neg\Leftrightarrow\}, \emptyset)$  by induction on LTL-formulas. Consider a formula  $\varphi \in \text{LTL}(\text{Op}_{\text{Un}}, \{\Leftrightarrow, \neg\Leftrightarrow\}, \{\mathbf{U}, \mathbf{R}\})$ :

- Assume that  $\varphi = x$  for any  $x \in \text{Prop}$ . We set  $\text{tr}(\varphi) := \varphi$  which satisfies conditions 1.-2.;

- Assume that  $\varphi = \neg\varphi'$  and that  $\text{tr}$  is defined on all sub-formulas of  $\varphi'$  and satisfies conditions 1.-2. on  $\varphi'$ . We set  $\text{tr}(\varphi) := \neg\text{tr}(\varphi')$ , which ensures that  $\text{SubF}(\text{tr}(\varphi)) = \{\neg\text{tr}(\varphi')\} \cup \text{SubF}(\text{tr}(\varphi')) \subseteq \text{tr}[\{\neg\varphi'\}] \cup \text{tr}[\text{SubF}(\varphi')] = \text{tr}[\text{SubF}(\varphi)]$ . Furthermore, for all  $\alpha \subseteq \text{Prop}$ , we have  $\alpha^\omega \models \text{tr}(\varphi)$  iff  $\alpha^\omega \not\models \text{tr}(\varphi')$  iff  $\alpha^\omega \not\models \varphi'$  iff  $\alpha^\omega \models \varphi$ . Thus,  $\varphi \equiv_1 \text{tr}(\varphi)$ ;
- For all  $\bullet \in \{\mathbf{X}, \mathbf{F}, \mathbf{G}\}$ , assume that  $\varphi = \bullet\varphi'$ , that  $\text{tr}$  is defined on all sub-formulas of  $\varphi'$  and satisfies conditions 1.-2. on  $\varphi'$ . Then, we set  $\text{tr}(\varphi) := \text{tr}(\varphi')$ , which ensures that  $\text{SubF}(\text{tr}(\varphi)) = \text{SubF}(\text{tr}(\varphi')) \subseteq \text{tr}[\text{SubF}(\varphi')] \subseteq \text{tr}[\text{SubF}(\varphi)]$ . Furthermore,  $\varphi \equiv_1 \text{tr}(\varphi)$  by Lemma 25 and since  $\varphi' \equiv_1 \text{tr}(\varphi')$ .
- For any  $\bullet \in \{\Leftrightarrow, \neg\Leftrightarrow, \mathbf{U}, \mathbf{R}\}$ , assume that  $\varphi = \varphi_1 \bullet \varphi_2$  and that  $\text{tr}$  is defined on all sub-formulas of  $\varphi_1$  and  $\varphi_2$  and satisfies conditions 1.-2. on  $\varphi_1$  and  $\varphi_2$ .
  - If  $\bullet \in \{\Leftrightarrow, \neg\Leftrightarrow\}$ , we set  $\text{tr}(\varphi) := \text{tr}(\varphi_1) \bullet \text{tr}(\varphi_2)$ , which ensures that  $\text{SubF}(\text{tr}(\varphi)) = \{\text{tr}(\varphi)\} \cup \text{SubF}(\text{tr}(\varphi_1)) \cup \text{SubF}(\text{tr}(\varphi_2)) \subseteq \text{tr}[\{\varphi\}] \cup \text{tr}[\text{SubF}(\varphi_1)] \cup \text{tr}[\text{SubF}(\varphi_2)] \subseteq \text{tr}[\text{SubF}(\varphi)]$ . Furthermore, since  $\varphi_1 \equiv_1 \text{tr}(\varphi_1)$  and  $\varphi_2 \equiv_1 \text{tr}(\varphi_2)$ , we have  $\varphi \equiv_1 \text{tr}(\varphi)$ ;
  - If  $\bullet \in \{\mathbf{U}, \mathbf{R}\}$ , we set  $\text{tr}(\varphi) := \text{tr}(\varphi_2)$ , which ensures that  $\text{SubF}(\text{tr}(\varphi)) = \text{SubF}(\text{tr}(\varphi_2)) \subseteq \text{tr}[\text{SubF}(\varphi_2)] \subseteq \text{tr}[\text{SubF}(\varphi)]$ . Furthermore,  $\varphi \equiv_1 \text{tr}(\varphi)$  by Lemma 25 and since  $\varphi_1 \equiv_1 \text{tr}(\varphi_1)$  and  $\varphi_2 \equiv_1 \text{tr}(\varphi_2)$

□

Now, in order to gain an intuition on the reduction that we will consider (and especially the problem from which we make that reduction), let us give the central property satisfied by  $\Leftrightarrow$ -formulas.

**Lemma 30.** *Consider a set of propositions  $\text{Prop}$ . Given an  $\Leftrightarrow$ -formula  $\varphi$ , we let  $\text{Neg}(\varphi) \in \mathbb{N}$  denote the number of occurrences of the  $\neg$  and  $\neg\Leftrightarrow$  operators in  $\varphi$ . Furthermore, given any subset  $\alpha \subseteq \text{Prop}$ , we also denote by  $\text{NbPr}_{\bar{\alpha}}(\varphi) \in \mathbb{N}$  the number of occurrences of propositions in  $\text{Prop} \setminus \alpha$  occurring in  $\text{Prop}$  (hence, some propositions may be counted several times if they appear more than once in  $\varphi$ ). Both of these numbers are defined inductively on the tree structure of the  $\Leftrightarrow$ -formula  $\varphi$ , without the notion of sub-formulas. Then, for all  $\alpha \subseteq \text{Prop}$ , we have:*

$$\alpha^\omega \models \varphi \text{ if and only if } \text{Neg}(\varphi) \text{ and } \text{NbPr}_{\bar{\alpha}}(\varphi) \text{ have the same parity}$$

*Stated algebraically, we have  $\alpha^\omega \models \varphi$  if and only if  $\text{Neg}(\varphi) + \text{NbPr}_{\bar{\alpha}}(\varphi) \pmod{2} = 0$ .*

*Proof.* We show by induction on  $\varphi$  the property  $\mathcal{P}(\varphi)$ :  $\alpha^\omega \models \varphi$  iff  $\text{Neg}(\varphi) + \text{NbPr}_{\bar{\alpha}}(\varphi)$  is even.

- If  $\varphi = x$  for some  $x \in \text{Prop}$ , then  $\alpha^\omega \models \varphi$  if and only if  $x \in \alpha$ , i.e.  $\text{NbPr}_{\bar{\alpha}}(\varphi) = 0$ . Since  $\text{Neg}(\varphi) = 0$  in any case,  $\mathcal{P}(\varphi)$  follows.
- Assume that  $\varphi = \neg\varphi'$  and that  $\mathcal{P}(\varphi')$  holds. Then,  $\text{Neg}(\varphi) = \text{Neg}(\varphi') + 1$  and  $\text{Prop}(\varphi) = \text{Prop}(\varphi')$ , thus  $\text{NbPr}_{\bar{\alpha}}(\varphi') = \text{NbPr}_{\bar{\alpha}}(\varphi)$ . Hence,  $\text{Neg}(\varphi')$  and  $\text{NbPr}_{\bar{\alpha}}(\varphi')$  have the same parity if and only if  $\text{Neg}(\varphi)$  and  $\text{NbPr}_{\bar{\alpha}}(\varphi)$  do not, and  $\mathcal{P}(\varphi)$  follows.
- Assume that  $\varphi = \varphi_1 \bullet \varphi_2$  for some  $\bullet \in \{\Leftrightarrow, \neg\Leftrightarrow\}$  and assume that both  $\mathcal{P}(\varphi_1)$  and  $\mathcal{P}(\varphi_2)$  hold. We have  $\text{NbPr}_{\bar{\alpha}}(\varphi) = \text{NbPr}_{\bar{\alpha}}(\varphi_1) + \text{NbPr}_{\bar{\alpha}}(\varphi_2)$ . Then:
  - If  $\bullet = \Leftrightarrow$ , we have  $\text{Neg}(\varphi) = \text{Neg}(\varphi_1) + \text{Neg}(\varphi_2)$ . Furthermore,  $\alpha^\omega \models \varphi$  iff we have  $\alpha^\omega \models \varphi_1$  and  $\alpha^\omega \models \varphi_2$  or  $\alpha^\omega \not\models \varphi_1$  and  $\alpha^\omega \not\models \varphi_2$ . That is, by  $\mathcal{P}(\varphi_1)$  and  $\mathcal{P}(\varphi_2)$ , we have  $\alpha^\omega \models \varphi$  iff

$$\text{Neg}(\varphi_1) + \text{NbPr}_{\bar{\alpha}}(\varphi_1) \pmod{2} = \text{Neg}(\varphi_2) + \text{NbPr}_{\bar{\alpha}}(\varphi_2) \pmod{2}$$

which is equivalent to

$$\text{Neg}(\varphi_1) + \text{Neg}(\varphi_2) + \text{NbPr}_{\bar{\alpha}}(\varphi_1) + \text{NbPr}_{\bar{\alpha}}(\varphi_2) = 0 \pmod{2}$$

That is:

$$\text{Neg}(\varphi) + \text{NbPr}_{\bar{\alpha}}(\varphi) = 0 \pmod{2}$$

- If  $\bullet = \neg \Leftrightarrow$ , we have  $\text{Neg}(\varphi) = \text{Neg}(\varphi_1) + \text{Neg}(\varphi_2) + 1$ . Furthermore,  $\alpha^\omega \models \varphi$  iff we have  $\alpha^\omega \models \varphi_1$  and  $\alpha^\omega \not\models \varphi_2$  or  $\alpha^\omega \models \varphi_2$  and  $\alpha^\omega \not\models \varphi_1$ . That is, by  $\mathcal{P}(\varphi_1)$  and  $\mathcal{P}(\varphi_2)$ , we have  $\alpha^\omega \models \varphi$  iff

$$\text{Neg}(\varphi_1) + \text{NbPr}_{\bar{\alpha}}(\varphi_1) \pmod{2} = \text{Neg}(\varphi_2) + \text{NbPr}_{\bar{\alpha}}(\varphi_2) + 1 \pmod{2}$$

which is equivalent to

$$\text{Neg}(\varphi_1) + \text{Neg}(\varphi_2) + 1 + \text{NbPr}_{\bar{\alpha}}(\varphi_1) + \text{NbPr}_{\bar{\alpha}}(\varphi_2) = 0 \pmod{2}$$

That is:

$$\text{Neg}(\varphi) + \text{NbPr}_{\bar{\alpha}}(\varphi) = 0 \pmod{2}$$

The property  $\mathcal{P}(\varphi)$  follows. □

We state below, as a corollary, the lemma above in a form that is easier to use for us.

**Corollary 31.** *Consider a set of propositions  $\text{Prop}$ . Given an  $\Leftrightarrow$ -formula  $\varphi$  and any subset  $\alpha \subseteq \text{Prop}$ , we let  $\text{TrueNbPr}_{\bar{\alpha}}(\varphi) \in \mathbb{N}$  denote the number of propositions in  $\text{Prop} \setminus \alpha$  that occur oddly many times in  $\varphi$ . Then, for all  $\alpha \subseteq \text{Prop}$ , we have:*

$$\alpha^\omega \models \varphi \text{ if and only if } \text{Neg}(\varphi) \text{ and } \text{TrueNbPr}_{\bar{\alpha}}(\varphi) \text{ have the same parity}$$

*Proof.* This is a direct consequence of Lemma 30 and the fact that  $\text{TrueNbPr}_{\bar{\alpha}}(\varphi)$  and  $\text{NbPr}_{\bar{\alpha}}(\varphi)$  have the same parity. □

This corollary suggests that the  $\text{LTL}_{\text{Learn}}(\{\neg\}, \{\Leftrightarrow, \neg \Leftrightarrow\}, \emptyset, \infty)$  learning problem is linked to modulo 2-calculus. In fact, there exists an NP-hard decision problem dealing with modulo 2-calculus, which we define below. The definition of this problem, and the proof that it is NP-complete can be found in [5].

**Definition 32** (Coset Weight). *We denote by CW the following decision problem:*

- *Input:*  $(A, k, y)$  where  $A$  is  $n \times l$ -matrix on  $\mathbb{Z}/2\mathbb{Z}$ ,  $k \leq l$  is an integer and  $y$  is an  $n$ -vector in  $\mathbb{Z}/2\mathbb{Z}$ .
- *Output:* yes iff there is an  $l$ -vector  $x$  in  $\mathbb{Z}/2\mathbb{Z}$  with at most  $k$  components of value 1 such that  $A \cdot x = y$  in  $\mathbb{Z}/2\mathbb{Z}$ .

*In terms of representation, the integer  $k$  may be given in unary.*

Since we are going to manipulate matrices, we introduce below the relevant notations.

**Definition 33.** *Given  $n, l \in \mathbb{N}_1$ , an  $n \times l$ -matrix  $A$  has  $n$  rows and  $l$  columns. Furthermore, for all  $1 \leq i \leq n$  and  $1 \leq j \leq l$ ,  $A_{i,j}$  refers to the coefficient of the matrix  $A$  at the intersection of the  $i$ -th row and  $j$ -th column.*

Let us now define the reduction that we consider.

**Definition 34.** Consider an instance  $(A, k, y)$  of the coset weight problem CW. If  $y = 0$  (i.e. it is the null vector), then  $(A, k, y)$  is trivially a positive instance of CW, hence in that case  $\text{In}_{(A, k, y)}^{\Leftrightarrow}$  is defined as any positive instance of the  $\text{LTL}_{\text{Learn}}(\{\neg\}, \{\Leftrightarrow, \neg\Leftrightarrow\}, \emptyset, \infty)$  decision problem. Otherwise, we define:

- $\text{Prop} := \{a_j \mid 1 \leq j \leq l\}$  to be the set of propositions;
- For all  $1 \leq i \leq n$ , we set  $\alpha_i := \{a_j \mid 1 \leq j \leq l, A_{i,j} = 0\}^\omega$ ;
- $\mathcal{P} := \{\beta\} \cup \{\alpha_i \mid 1 \leq i \leq n, y[i] = 0\}$  with  $\beta := \text{Prop}^\omega$ ;
- $\mathcal{N} := \{\alpha_i \mid 1 \leq i \leq n, y[i] = 1\}$ ;
- $B = 2k$ .

Then, we define the input  $\text{In}_{(A, k, y)}^{\Leftrightarrow} := (\text{Prop}, \mathcal{P}, \mathcal{N}, B)$  of the  $\text{LTL}_{\text{Learn}}(\{\neg\}, \{\Leftrightarrow, \neg\Leftrightarrow\}, \emptyset, \infty)$  decision problem.

The definition above satisfies the lemma below.

**Lemma 35.** Consider a set  $\mathbf{B}^l$  of logical binary operators and assume that  $\emptyset \neq \mathbf{B}^l \subseteq \{\Leftrightarrow, \neg\Leftrightarrow\}$ . Then, for all  $\mathbf{U}^t \subseteq \text{Op}_{\mathbf{U}^n}$  and  $\mathbf{B}^t \subseteq \{\mathbf{U}, \mathbf{W}\}$ ,  $(A, k, y)$  is a positive instance of the coset weight problem CW if and only if  $\text{In}_{(A, k, y)}^{\Leftrightarrow}$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem.

*Proof.* If  $y = 0$ , the equivalence is straightforward. Let us now assume in the following that  $y \neq 0$ . In that case, we have  $\mathcal{P} \neq \emptyset$  and  $\mathcal{N} \neq \emptyset$ .

Assume that  $(A, k, y)$  is a positive instance of the coset weight problem CW and consider an  $l$ -vector  $x$  in  $\mathbb{Z}/2\mathbb{Z}$  with at most  $k$  components of value 1 such that  $Ax = y$ . Since  $y \neq 0$ ,  $x$  has at least one component of value 1. We let  $H := \{a_j \mid 1 \leq j \leq l, x[j] = 1\}$ . We write  $H = \{a_{j_1}, \dots, a_{j_r}\}$  with  $1 \leq r \leq k$ . We then consider the formulas:

- $\varphi_{\Leftrightarrow} := a_{j_1} \Leftrightarrow a_{j_2} \Leftrightarrow \dots \Leftrightarrow a_{j_r}$
- We let  $\varphi_{\neg\Leftrightarrow}^{\text{even}} := a_{j_1} \neg\Leftrightarrow a_{j_2} \neg\Leftrightarrow \dots \neg\Leftrightarrow a_{j_r}$ . We also define:

$$\varphi_{\neg\Leftrightarrow} := \begin{cases} \varphi_{\neg\Leftrightarrow}^{\text{even}} & \text{if } r \text{ is even} \\ \neg\varphi_{\neg\Leftrightarrow}^{\text{even}} & \text{if } r \text{ is odd} \end{cases}$$

Both of these formulas have size at most  $2r \leq 2k = B$ .

By definition, we have both  $\text{Neg}(\varphi_{\Leftrightarrow})$  and  $\text{Neg}(\varphi_{\neg\Leftrightarrow})$  even. Therefore, by Corollary 31, both  $\varphi_{\Leftrightarrow}$  and  $\varphi_{\neg\Leftrightarrow}$  accept the word  $\beta$ . Consider now some  $1 \leq i \leq n$ .

- Assume that  $y[i] = 0$ . Then, they are evenly many indices  $j \in [1, \dots, l]$  such that  $x[j] = 1 = A_{i,j}$ . That is,  $|H \setminus \alpha_i|$  is even. Hence, by Corollary 31,  $\varphi_{\Leftrightarrow}, \varphi_{\neg\Leftrightarrow}$  accept  $\alpha_i \in \mathcal{P}$ .
- Assume that  $y[i] = 1$ . Then, they are oddly many indices  $j \in [1, \dots, l]$  such that  $x[j] = 1 = A_{i,j}$ . That is,  $|H \setminus \alpha_i|$  is odd. Hence, by Corollary 31,  $\varphi_{\Leftrightarrow}, \varphi_{\neg\Leftrightarrow}$  reject  $\alpha_i \in \mathcal{N}$ .

Hence,  $\varphi_{\Leftrightarrow}, \varphi_{\neg\Leftrightarrow}$  accept  $\mathcal{P}$  and reject  $\mathcal{N}$ . Thus,  $\text{In}_{(A,k,y)}^{\Leftrightarrow}$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem.

Assume now that  $\text{In}_{(A,k,y)}^{\Leftrightarrow}$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem. Consider an  $\text{LTL}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l)$ -formula  $\varphi$  of size at most  $B$  that accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . By Lemma 29, there is a  $\Leftrightarrow$ -formula  $\varphi'$  of size at most  $B$  that is equivalent to the formula  $\varphi$  on size-1 infinite words. By Corollary 31, since  $\varphi'$  accepts  $\beta$ , it follows that  $\text{Neg}(\varphi')$  is even. We let  $H := \{i \in [1, \dots, l] \mid a_i \text{ occurs oddly many times in } \varphi'\}$ . By Lemma 12, we have  $H \leq k$ . We define the  $l$ -vector  $x$  in  $\mathbb{Z}/2\mathbb{Z}$  by, for all  $1 \leq j \leq l$ ,  $x[j] := 1$  if and only if  $j \in H$ . Consider now any  $1 \leq i \leq n$ .

- If  $\alpha_i \in \mathcal{P}$ , we have  $|H \setminus \alpha_i[1]|$  even, by Corollary 31. In addition, for all  $j \in H \setminus \alpha_i[1]$ , we have  $x[j] = 1 = A_{i,j}$  whereas, for all  $j \notin H \setminus \alpha_i$ , we have  $x[j] = 0$  or  $A_{i,j} = 0$ . Therefore,  $Ax[i] = 0 = y[i]$ .
- If  $\alpha_i \in \mathcal{N}$ , we have  $|H \setminus \alpha_i|$  odd, by Corollary 31. Therefore,  $Ax[i] = 1 = y[i]$ .

Hence,  $(A, k, y)$  is a positive instance of the coset weight problem CW.  $\square$

We deduce the corollary below.

**Corollary 36.** *Consider a set  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  of binary logical operators with  $\mathbf{B}^l \cap \{\Leftrightarrow, \neg\Leftrightarrow\} \neq \emptyset$ . For all  $\mathbf{U}^t \subseteq \text{Op}_{\text{Un}}$  and  $\mathbf{B}^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$ , the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem is NP-hard.*

*Proof.* If we have  $\mathbf{B}^l \cap \{\vee, \Rightarrow, \Leftarrow, \wedge, \neg\Rightarrow, \neg\Leftarrow, \neg\vee, \neg\wedge\} \neq \emptyset$ , Corollary 17 or Corollary 23 gives that the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem is NP-hard. Furthermore, if  $\mathbf{B}^t \cap \{\mathbf{W}, \mathbf{M}\} \neq \emptyset$ , then Corollary 27 gives that the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem is NP-hard. If it is not the case, then we have both  $\emptyset \neq \mathbf{B}^l \subseteq \{\Leftrightarrow, \neg\Leftrightarrow\}$  and  $\mathbf{B}^t \subseteq \{\mathbf{U}, \mathbf{W}\}$ . The result then follows from Lemma 35, the fact that the Coset Weight decision problem is NP-hard and the fact that the instance  $\text{In}_{(l,C,k)}^{\Leftrightarrow}$  can be computed in logarithmic space from  $(l, C, k)$ .  $\square$

### 3.2.6 Proof of Theorem 10: with the temporal operators $\mathbf{U}$ and $\mathbf{R}$

We conclude with the temporal operators  $\mathbf{U}$  and  $\mathbf{R}$ . We will consider a reduction from the hitting set problem again. However, as can be seen in Lemma 25, with these operators, a reduction with size-1 infinite words will not work since they simplify too much on them.

We define below a way to build infinite words or LTL-formulas from subsets of integers.

**Definition 37.** *Let  $l \in \mathbb{N}$ . We consider a set of propositions  $\text{Prop}^l := \{a_{i,j}, b_{i,j} \mid 1 \leq i \leq j \leq l\}$ . Consider any set subset  $C \subseteq [1, \dots, l]$ . For all  $1 \leq i \leq l$ , we let  $\mathbf{S}_i(C) := \{(k, j) \mid 1 \leq k \leq i, i \leq j, j \in C\}$ . Then, we let:*

$$w^{\mathbf{R}}(C) := w_1(C) \cdot \dots \cdot w_l(C) \cdot \emptyset^\omega \in (2^{\text{Prop}^l})^\omega$$

and

$$w^{\mathbf{U}}(C) := w_1(C) \cdot \dots \cdot w_l(C) \cdot (\text{Prop}^l)^\omega \in (2^{\text{Prop}^l})^\omega$$

where, for all  $1 \leq i \leq l$ , we have:

$$w_i(C) := \{a_{k,j} \mid (k, j) \in \mathbf{S}_i(C)\} \cup \{b_{k,j} \mid (k, j) \notin \mathbf{S}_i(C)\}$$

Furthermore, consider any  $H = \{i_1, \dots, i_r\} \subseteq [1, \dots, l]$  with  $i_1 < i_2 < i_3 < \dots < i_r$ . We let  $i_0 := 1$  and we define:

- $\varphi_{\mathbf{R}}(H, r) := a_{i_{r-1}, i_r}$ , and  $\varphi_{\mathbf{U}}(H, r) := b_{i_{r-1}, i_r}$ ;

- for all  $2 \leq p \leq r$ ,  $\varphi_{\mathbf{R}}(H, p-1) := \varphi_{\mathbf{R}}(H, p) \mathbf{R} a_{i_{p-2}, i_{p-1}}$ , and  $\varphi_{\mathbf{U}}(H, p-1) := \varphi_{\mathbf{U}}(H, p) \mathbf{U} b_{i_{p-2}, i_{p-1}}$ .

We let  $\varphi_{\mathbf{R}}(H) := \varphi_{\mathbf{R}}(H, 1)$  and  $\varphi_{\mathbf{U}}(H) := \varphi_{\mathbf{U}}(H, 1)$ . That way, we have:

$$\varphi_{\mathbf{R}}(H) = ((a_{i_{r-1}, i_r} \mathbf{R} a_{i_{r-2}, i_{r-1}}) \mathbf{R} \dots) \mathbf{R} a_{1, i_1}$$

and

$$\varphi_{\mathbf{U}}(H) = ((b_{i_{r-1}, i_r} \mathbf{U} b_{i_{r-2}, i_{r-1}}) \mathbf{U} \dots) \mathbf{U} b_{1, i_1}$$

We first establish below what the  $\mathbf{R}$ -formula satisfies.

**Lemma 38.** Consider any  $H = \{i_1, \dots, i_r\} \subseteq [1, \dots, l]$  with  $i_1 < i_2 < i_3 < \dots < i_r$  and some  $C \subseteq [1, \dots, l]$ . We have the following equivalence:

$$H \subseteq C \text{ if and only if } w^{\mathbf{R}}(C) \models \varphi_{\mathbf{R}}(H)$$

*Proof.* For all  $r \geq p \geq 1$ , we let  $H_p := \{i_p, \dots, i_r\} \subseteq H$ . Let us show by induction on  $r \geq p \geq 1$  the property  $\mathcal{P}(p)$ :

For all  $1 \leq i \leq i_{p-1} - 1$ :  $w^{\mathbf{R}}(C)[i:] \not\models \varphi_{\mathbf{R}}(H, p)$  and  $w^{\mathbf{R}}(C)[i_{p-1}:] \models \varphi_{\mathbf{R}}(H, p)$  iff  $H_p \subseteq C$

We start with the base case  $\mathcal{P}(r)$ . For all  $1 \leq i \leq i_{r-1} - 1$ , we have  $a_{i_{r-1}, i_r} \notin w^{\mathbf{R}}(C)[i]$ . Furthermore, we have  $a_{i_{r-1}, i_r} \in w^{\mathbf{R}}(C)[i_{r-1}]$  if and only if  $i_r \in C$ . Since  $\varphi_{\mathbf{R}}(H, r) = a_{i_{r-1}, i_r}$ ,  $\mathcal{P}(r)$  follows.

Let us now assume that  $\mathcal{P}(p)$  holds for some  $2 \leq p \leq r$ . We have:

$$\varphi_{\mathbf{R}}(H, p-1) = \varphi_{\mathbf{R}}(H, p) \mathbf{R} a_{i_{p-2}, i_{p-1}}$$

For all  $1 \leq i \leq i_{p-2} - 1$ , we have  $a_{i_{p-2}, i_{p-1}} \notin w^{\mathbf{R}}(C)[i]$ , thus  $w^{\mathbf{R}}(C)[i:] \not\models a_{i_{p-2}, i_{p-1}}$ . In addition, by  $\mathcal{P}(p)$ , for all  $1 \leq i \leq i_{p-1} - 1$ :  $w^{\mathbf{R}}(C)[i:] \not\models \varphi_{\mathbf{R}}(H, p)$  (in particular, this holds for  $i = i_{p-2}$ ). It follows that, for all  $1 \leq i \leq i_{p-2} - 1$ , we have  $w^{\mathbf{R}}(C)[i:] \not\models \varphi_{\mathbf{R}}(H, p-1)$ .

- Assume that  $H_{p-1} \subseteq C$ . Then, we also have  $H_p \subseteq C$ , thus, by  $\mathcal{P}(p)$ , we have  $w^{\mathbf{R}}(C)[i_{p-1}:] \models \varphi_{\mathbf{R}}(H, p)$ . Furthermore, for all  $i_{p-2} \leq i \leq i_{p-1}$ , we have  $w^{\mathbf{R}}(C)[i] \models a_{i_{p-2}, i_{p-1}}$  since  $i_{p-1} \in C$ . It follows that:

$$w^{\mathbf{R}}(C)[i_{p-2}:] \models \varphi_{\mathbf{R}}(H, p-1)$$

- Assume that  $H_{p-1} \not\subseteq C$ . Then, there are two cases, since  $H_{p-1} = H_p \cup \{i_{p-1}\}$ .
  - Assume that  $H_p \not\subseteq C$ . By  $\mathcal{P}(p)$ , for all  $1 \leq i \leq i_{p-1}$ , we have  $w^{\mathbf{R}}(C)[i:] \not\models \varphi_{\mathbf{R}}(H, p)$ . Furthermore,  $w^{\mathbf{R}}(C)[i_{p-1} + 1:] \not\models a_{i_{p-2}, i_{p-1}}$ . Therefore,  $w^{\mathbf{R}}(C)[i_{p-2}:] \not\models \varphi_{\mathbf{R}}(H, p-1)$ .
  - Assume that  $i_{p-1} \notin C$ . Then, for all  $1 \leq i \leq l$ , we have  $w^{\mathbf{R}}(C)[i:] \not\models a_{i_{p-2}, i_{p-1}}$ . Furthermore, by  $\mathcal{P}(p)$ , for all  $1 \leq i \leq i_{p-1} - 1$ :  $w^{\mathbf{R}}(C)[i:] \not\models \varphi_{\mathbf{R}}(H, p)$ . It follows that  $w^{\mathbf{R}}(C)[i_{p-2}:] \not\models \varphi_{\mathbf{R}}(H, p-1)$ .

Hence, the property  $\mathcal{P}(p-1)$  holds. Therefore,  $\mathcal{P}(p)$  holds for all  $1 \leq p \leq r$ . The lemma is then given by  $\mathcal{P}(1)$ , since  $H_1 = H$ ,  $w^{\mathbf{R}}(C)[1:] = w^{\mathbf{R}}(C)$  and  $\varphi_{\mathbf{R}}(H, 1) = \varphi_{\mathbf{R}}(H)$ .  $\square$

Let us now consider the case of  $\mathbf{U}$ -formulas.



**Lemma 39.** Consider any  $H = \{i_1, \dots, i_r\} \subseteq [1, \dots, l]$  with  $i_1 < i_2 < i_3 < \dots < i_r$  and some  $C \subseteq [1, \dots, l]$ . We have the following equivalence:

$$H \subseteq C \text{ if and only if } w^{\mathbf{U}}(C) \not\models \varphi_{\mathbf{U}}(H)$$

*Proof.* Let us show by induction on  $r \geq p \geq 1$  the property  $\mathcal{P}(p)$ : for all  $k \in \mathbb{N}_1$ , we have  $w^{\mathbf{U}}(C)[k:] \models \varphi_{\mathbf{U}}(H, p)$  if and only if  $w^{\mathbf{R}}(C)[k:] \not\models \varphi_{\mathbf{R}}(H, p)$ .

Let us start with the base case. We have  $\varphi_{\mathbf{U}}(H, r) = b_{i_{r-1}, i_r}$  and  $\varphi_{\mathbf{R}}(H, r) = a_{i_{r-1}, i_r}$ . Furthermore, for all  $k \in \mathbb{N}_1$ , we have  $a_{i_{r-1}, i_r} \in w^{\mathbf{U}}(C)[k]$  if and only if  $b_{i_{r-1}, i_r} \notin w^{\mathbf{R}}(C)[k]$ . Thus,  $\mathcal{P}(r)$  follows.

Assume now that  $\mathcal{P}(p)$  holds for some  $2 \leq p \leq r$ . We have:

$$\varphi_{\mathbf{U}}(H, p-1) = \varphi_{\mathbf{U}}(H, p) \ \mathbf{U} \ b_{i_{p-2}, i_{p-1}}$$

and

$$\varphi_{\mathbf{R}}(H, p-1) = \varphi_{\mathbf{R}}(H, p) \ \mathbf{R} \ a_{i_{p-2}, i_{p-1}}$$

Thus, by definition of the operator  $\mathbf{R}$ , we have  $\varphi_{\mathbf{R}}(H, p-1)$  equivalent to  $\neg(\neg\varphi_{\mathbf{R}}(H, p) \ \mathbf{U} \ \neg a_{i_{p-2}, i_{p-1}})$ . In addition, for all  $i \in \mathbb{N}_1$ , we have  $b_{i_{p-2}, i_{p-1}} \in w^{\mathbf{U}}(C)[i]$  iff  $a_{i_{p-2}, i_{p-1}} \in w^{\mathbf{R}}(C)[i]$ . Thus, by  $\mathcal{P}(p)$ , for all  $k \in \mathbb{N}_1$ , we have:

$$\begin{aligned} w^{\mathbf{U}}(C)[k:] \models \varphi_{\mathbf{U}}(H, p-1) &\text{ iff } \exists j \geq k, b_{i_{p-2}, i_{p-1}} \in w^{\mathbf{U}}(C)[j] \text{ and } \forall k \leq i < j, w^{\mathbf{U}}(C)[i:] \models \varphi_{\mathbf{U}}(H, p) \\ &\text{ iff } \exists j \geq k, a_{i_{p-2}, i_{p-1}} \notin w^{\mathbf{R}}(C)[j] \text{ and } \forall k \leq i < j, w^{\mathbf{R}}(C)[i:] \not\models \varphi_{\mathbf{R}}(H, p) \\ &\text{ iff } w^{\mathbf{R}}(C)[k:] \models \neg\varphi_{\mathbf{R}}(H, p) \ \mathbf{U} \ \neg a_{i_{p-2}, i_{p-1}} \\ &\text{ iff } w^{\mathbf{R}}(C)[k:] \models \neg\varphi_{\mathbf{R}}(H, p-1) \end{aligned}$$

Thus, the property  $\mathcal{P}(p-1)$  holds. In fact, it holds for all  $1 \leq p \leq r$ . This lemma is then a direct consequence of the property  $\mathcal{P}(1)$  and Lemma 39.  $\square$

We can now define the reduction that we consider.

**Definition 40.** Consider an instance  $(l, C, k)$  of the hitting set problem  $\text{Hit}$ . We define:

- $\text{Prop} := \text{Prop}^l = \{a_{i,j}, b_{i,j} \mid 1 \leq i \leq j \leq l\}$  as set of propositions;
- $\mathcal{P}^{\mathbf{U}} := \{v_1^{\mathbf{U}}, \dots, v_n^{\mathbf{U}}\}$  and  $\mathcal{P}^{\mathbf{R}} := \{v^{\mathbf{R}}\}$ ;
- $\mathcal{N}^{\mathbf{U}} := \{v^{\mathbf{U}}\}$  and  $\mathcal{N}^{\mathbf{R}} := \{v_1^{\mathbf{R}}, \dots, v_n^{\mathbf{R}}\}$ ;
- $B = 2k - 1$ .

with, for  $\bullet \in \{\mathbf{R}, \mathbf{U}\}$ , we let  $v^\bullet := w^\bullet([1, \dots, l]) \in (2^{\text{Prop}})$  and for all  $1 \leq i \leq n$ , we let  $v_i^\bullet := w^\bullet([1, \dots, l] \setminus C_i) \in (2^{\text{Prop}})$ .

Then, we define the inputs  $\text{In}_{(l,C,k)}^{\mathbf{R}} := (\text{Prop}, \mathcal{P}^{\mathbf{R}}, \mathcal{N}^{\mathbf{R}}, B)$  and  $\text{In}_{(l,C,k)}^{\mathbf{U}} := (\text{Prop}, \mathcal{P}^{\mathbf{U}}, \mathcal{N}^{\mathbf{U}}, B)$  of the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem.

This definition satisfies the lemma below.

**Lemma 41.** Let  $\bullet \in \{\mathbf{R}, \mathbf{U}\}$ . Consider a set  $\mathbf{B}^t$  of operators and assume that  $\bullet \in \mathbf{B}^t \neq \emptyset$ . Then, for all  $\mathbf{U}^t \subseteq \text{Op}_{\mathbf{U}^t}$  and  $\mathbf{B}^l \subseteq \text{Op}_{\mathbf{B}^l}^{\text{lg}}$ ,  $(l, C, k)$  is a positive instance of the hitting set problem  $\text{Hit}$  if and only if  $\text{In}_{(l,C,k)}^\bullet$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem.

*Proof.* Assume that  $(l, C, k)$  is a positive instance of the hitting set problem Hit. Consider a hitting set  $H \subseteq [1, \dots, l]$ . We consider the formulas  $\varphi_{\mathbf{R}} := \varphi_{\mathbf{R}}(H)$  and  $\varphi_{\mathbf{U}} := \varphi_{\mathbf{U}}(H)$ . Clearly, these formulas have size  $2k - 1 = B$ . In addition, Lemmas 38 and 39 gives that  $v^{\mathbf{U}} \models \varphi_{\mathbf{U}}$  and  $v^{\mathbf{R}} \models \varphi_{\mathbf{R}}$ . Consider now some  $1 \leq i \leq n$ . Since  $H \cap C_i \neq \emptyset$ , it follows that  $H \not\subseteq [1, \dots, l] \setminus C_i$ , hence, by Lemmas 38 and 39, we have  $v_i^{\mathbf{U}} \models \varphi_{\mathbf{U}}$  and  $v_i^{\mathbf{R}} \not\models \varphi_{\mathbf{R}}$ . Therefore, for  $\bullet \in \{\mathbf{R}, \mathbf{U}\}$ , we have  $\text{In}_{(l, C, k)}^{\bullet}$  a positive instance of the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem.

Assume now that  $\text{In}_{(l, C, k)}^{\bullet}$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem. Consider an LTL-formula  $\varphi$  of size at most  $B = 2k - 1$  that accepts  $\mathcal{P}^{\bullet}$  and rejects  $\mathcal{N}^{\bullet}$ . We let  $H := \{\alpha \in [1, \dots, l] \mid \exists 1 \leq i \leq l, \{a_{i, \alpha}, b_{i, \alpha}\} \cap \text{Prop}(\varphi) \neq \emptyset\}$ . By Lemma 12, we have  $|H| \leq k$ . Let us show that it is a hitting set. Consider some  $1 \leq p \leq n$ . Given two sets  $A$  and  $B$ , we let  $A \Delta B$  denote the symmetric difference:  $A \Delta B := A \setminus B \cup B \setminus A$ . Then, we have that:

$$\forall 1 \leq i \leq l, v_p^{\bullet}[i] \Delta v^{\bullet}[i] = \{a_{k, j}, b_{k, j} \mid 1 \leq k \leq i, i \leq j, j \in C_p\}$$

and

$$\forall i \geq l, v_p^{\bullet}[i] = v^{\bullet}[i]$$

Hence, by Lemma 11, it follows that  $\text{Prop}(\varphi)$  must contain at least a variable  $a_{k, j}$  or  $b_{k, j}$  with  $j \in C_p$ . That is,  $C_p \cap H \neq \emptyset$ . Since this holds for all  $1 \leq p \leq n$ , it follows that  $H$  is a hitting set and  $(l, C, k)$  is a positive instance of the hitting set problem Hit.  $\square$

We obtain the corollary below.

**Corollary 42.** *Consider a set  $\mathbf{B}^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$  of binary temporal operators with  $\mathbf{B}^t \cap \{\mathbf{U}, \mathbf{R}\} \neq \emptyset$ . For all  $\mathbf{U}^t \subseteq \text{Op}_{\text{Un}}$  and  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$ , the  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, \infty)$  decision problem is NP-hard.*

*Proof.* This is a direct consequence of Lemma 41 and the fact that the instances  $\text{In}_{(l, C, k)}^{\mathbf{U}}$  and  $\text{In}_{(l, C, k)}^{\mathbf{R}}$  can be computed in logarithmic space from  $(l, C, k)$ .  $\square$

The proof of Theorem 10 follows.

*Proof.* It is a direct consequence of Corollaries 17, 23, 36, 27 and 42.  $\square$

### 3.3 CTL learning is at least as hard as LTL learning

Let us now turn to CTL learning. Our goal is to show that CTL learning is at least as hard as LTL learning, under logarithmic space reductions, regardless of the operators allowed. This is formally stated in the theorem below.

**Theorem 43.** *For all  $\mathbf{U}^t \subseteq \text{Op}_{\text{Un}}$ ,  $\mathbf{B}^t \subseteq \text{Op}_{\text{Bin}}^{\text{tp}}$ ,  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N} \cup \{\infty\}$ , the decision problem  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, n)$  is at least as hard as the decision problem  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l)$ , under logarithmic space reduction.*

*Therefore, it is also the case of the decision problems  $\text{ATL}_{\text{Learn}}^k(\mathbf{U}^t, \mathbf{B}^t, \mathbf{B}^l, n)$ , for  $k \in \mathbb{N}_1$ .*

The proof of this theorem consists in translating ultimately periodic words into Kripke structures, translating LTL-formulas into CTL-formulas, and vice versa, and relating all these translations together.

Let first translate ultimately periodic words into Kripke structure of the same size.

**Definition 44.** *Consider a set of propositions Prop and an ultimately periodic word  $w = u \cdot v^{\omega} \in (2^{\text{Prop}})^{\omega}$ . We  $k := |u|$  and  $n := |v|$ . Then, we define the Kripke structure  $K_w = \langle Q, I, \text{Prop}, \pi, \text{Succ} \rangle$  where:*

- $Q := \{q_1^u, \dots, q_k^u, q_1^v, \dots, q_n^v\}$ ;

- $I := \{q_1^u\}$ ;
- for all  $1 \leq i \leq k-1$ , we have  $\text{Succ}(q_i^u) := \{q_{i+1}^u\}$ ,  $\text{Succ}(q_k^u) := \{q_1^v\}$  and for all  $1 \leq i \leq n-1$ , we have  $\text{Succ}(q_i^v) := \{q_{i+1}^v\}$ ,  $\text{Succ}(q_n^v) := \{q_1^v\}$ ;
- for all  $1 \leq i \leq k$ ,  $\pi(q_i^u) := u[i] \subseteq \text{Prop}$  and for all  $1 \leq i \leq n$ ,  $\pi(q_i^v) := v[i] \subseteq \text{Prop}$ .

We then define below how to translate CTL-formulas into LTL-formulas and vice versa, while staying in the same fragment.

**Definition 45.** Consider a set of propositions  $\text{Prop}$  and  $U^t \subseteq \text{Op}_{U^n}$ ,  $B^t \subseteq \text{Op}_{B_{\text{Bin}}^{\text{tp}}}$  and  $B^l \subseteq \text{Op}_{B_{\text{Bin}}^{\text{lg}}}$ . Then, we let  $f_{\text{C-L}(U^t, B^t, B^l)} : \text{CTL}(U^t, B^t, B^l) \rightarrow \text{LTL}(U^t, B^t, B^l)$  be such that it removes all  $\exists$  or  $\forall$  quantifiers from a CTL-formula, while keeping the same operators. More formally:

- $f_{\text{C-L}(U^t, B^t, B^l)}(p) = p$  for all  $p \in \text{Prop}$ ;
- $f_{\text{C-L}(U^t, B^t, B^l)}(\neg\phi) = \neg f_{\text{C-L}(U^t, B^t, B^l)}(\phi)$ ;
- $f_{\text{C-L}(U^t, B^t, B^l)}(\diamond(\bullet f_{\text{C-L}(U^t, B^t, B^l)}(\phi))) = \bullet f_{\text{C-L}(U^t, B^t, B^l)}(\phi)$  for all  $\bullet \in U^t \setminus \{\neg\}$  and  $\diamond \in \{\exists, \forall\}$ ;
- $f_{\text{C-L}(U^t, B^t, B^l)}(\phi_1 \bullet \phi_2) = f_{\text{C-L}(U^t, B^t, B^l)}(\phi_1) \bullet f_{\text{C-L}(U^t, B^t, B^l)}(\phi_2)$  for all  $\bullet \in B^l$ ;
- $f_{\text{C-L}(U^t, B^t, B^l)}(\diamond(\phi_1 \bullet \phi_2)) = f_{\text{C-L}(U^t, B^t, B^l)}(\phi_1) \bullet f_{\text{C-L}(U^t, B^t, B^l)}(\phi_2)$  for all  $\bullet \in B^t$  and  $\diamond \in \{\exists, \forall\}$ .

We also define the function  $f_{\text{L-C}(U^t, B^t, B^l)} : \text{LTL}(U^t, B^t, B^l) \rightarrow \text{CTL}(U^t, B^t, B^l)$  that adds an  $\exists$  quantifier before every temporal operator of an LTL-formula, thus transforming it into a CTL-formula. More formally:

- $f_{\text{L-C}(U^t, B^t, B^l)}(p) = p$  for all  $p \in \text{Prop}$ ;
- $f_{\text{L-C}(U^t, B^t, B^l)}(\neg\varphi) = \neg f_{\text{L-C}(U^t, B^t, B^l)}(\varphi)$ ;
- $f_{\text{L-C}(U^t, B^t, B^l)}(\bullet f_{\text{L-C}(U^t, B^t, B^l)}(\varphi)) = \exists(\bullet f_{\text{L-C}(U^t, B^t, B^l)}(\varphi))$  for all  $\bullet \in U^t \setminus \{\neg\}$ ;
- $f_{\text{L-C}(U^t, B^t, B^l)}(\varphi_1 \bullet \varphi_2) = f_{\text{L-C}(U^t, B^t, B^l)}(\varphi_1) \bullet f_{\text{L-C}(U^t, B^t, B^l \circ_U, \circ_B)}(\varphi_2)$  for all  $\bullet \in B^l$ ;
- $f_{\text{L-C}(U^t, B^t, B^l)}(\varphi_1 \bullet \varphi_2) = \exists(f_{\text{L-C}(U^t, B^t, B^l)}(\varphi_1) \bullet f_{\text{L-C}(U^t, B^t, B^l)}(\varphi_2))$  for all  $\bullet \in B^t$ .

A direct proof by induction shows that the above definition satisfies the proposition below:

**Proposition 46.** Consider a set of propositions  $\text{Prop}$  and  $U^t \subseteq \text{Op}_{U^n}$ ,  $B^t \subseteq \text{Op}_{B_{\text{Bin}}^{\text{tp}}}$  and  $B^l \subseteq \text{Op}_{B_{\text{Bin}}^{\text{lg}}}$ . We have:

- for all CTL-formulas  $\phi \in \text{CTL}(U^t, B^t, B^l)$ ,  $|\phi| \geq |f_{\text{C-L}(U^t, B^t, B^l)}(\phi)|$  (the size may decrease for instance for the CTL-formula  $\phi = \exists \mathbf{X} p \vee \forall \mathbf{X} p$ );
- for all LTL-formulas  $\varphi \in \text{LTL}(U^t, B^t, B^l)$ ,  $|\varphi| = |f_{\text{L-C}(U^t, B^t, B^l)}(\varphi)|$ ;
- for all LTL-formulas  $\varphi \in \text{LTL}(U^t, B^t, B^l)$ , we have  $f_{\text{C-L}(U^t, B^t, B^l)}(f_{\text{L-C}(U^t, B^t, B^l)}(\varphi)) = \varphi$ .

Interestingly for us, Definition 45 also satisfies the lemma below, which is slightly less direct to show that Proposition 46 above.

**Lemma 47.** Consider a set of propositions  $\text{Prop}$  and  $U^t \subseteq \text{Op}_{U^n}$ ,  $B^t \subseteq \text{Op}_{B_{\text{Bin}}^{\text{tp}}}$  and  $B^l \subseteq \text{Op}_{B_{\text{Bin}}^{\text{lg}}}$ . Let  $\phi \in \text{CTL}(U^t, B^t, B^l)$  be a CTL-formula. For all ultimately periodic words  $w = u \cdot v^\omega \in 2^{\text{Prop}}$ , we have  $K_w \models \phi$  if and only if  $w \models f_{\text{C-L}(U^t, B^t, B^l)}(\phi)$ .

*Proof.* Let  $k := |u|$  and  $n := |v|$ . In the Kripke structure  $K_w$ , all states have exactly one successor. Hence, for all states  $q \in Q$ , we have  $|\text{Out}^Q(q)| = 1$  and we let  $\rho_q \in Q^\omega$  be the infinite path such that  $\text{Out}^Q(q) = \{\rho_q\}$ .

Let  $g : \mathbb{N}_1 \rightarrow Q$  be such that for all  $i \in \mathbb{N}_1$  we have:

$$g(i) := \begin{cases} q_i^u & \text{if } i \leq k \\ q_{1+(i-k-1 \bmod n)}^v & \text{if } i \geq k+1 \end{cases}$$

For all  $i \in \mathbb{N}_1$ , we have  $\text{Succ}(g(i)) = \{g(i+1)\}$ . Indeed, for all  $i \in \mathbb{N}_1$ , we have:

- if  $i \leq k-1$ , then  $g(i) = q_i^u$  and  $g(i+1) = q_{i+1}^u$ ;
- $g(k) = q_k^u$  and  $g(k+1) = q_1^v$ ;
- if  $i \geq k+1$  and  $l := (i-k-1) \bmod n \leq n-2$ , then  $g(i) = q_{l+1}^v$  and  $g(i+1) = q_{l+2}^v$ ;
- if  $i \geq k+1$  and  $(i-k-1) \bmod n = n-1$ , then  $g(i) = q_n^v$  and  $g(i+1) = q_1^v$ .

It follows that, for all  $i \in \mathbb{N}_1$ , we have  $w[i:] = \pi(\rho_{g(i)}) \in (2^{\text{Prop}})^\omega$ .

Now, for all  $i \in \mathbb{N}_1$ , we denote by  $K_w^i$  the Kripke structure that is equal to the Kripke structure  $K_w$  except that the initial state is now  $g(i)$ , i.e.  $I = \{g(i)\}$ . Let us show by induction on CTL-formulas  $\phi \in \text{CTL}(\text{U}^t, \text{B}^t, \text{B}^l)$  the property  $\mathcal{P}(\phi)$ : for all  $i \in \mathbb{N}_1$ , we have  $K_w^i \models \phi$  if and only if  $w[i:] \models \mathbf{f}_{\text{C-L}(\text{U}^t, \text{B}^t, \text{B}^l)}(\phi)$ . Consider some  $\phi \in \text{CTL}(\text{U}^t, \text{B}^t, \text{B}^l)$ . We have:

- Assume that  $\phi = x$  for any  $x \in \text{Prop}$ . In that case,  $\mathbf{f}_{\text{C-L}(\text{U}^t, \text{B}^t, \text{B}^l)}(\phi) = x$ . Let  $i \in \mathbb{N}_1$ . We have  $K_w^i \models \phi$  iff  $x \in \pi(q_{g(i)}) = w[i]$  iff  $w[i:] \models \phi$ . Hence,  $\mathcal{P}(\phi)$  holds;
- Assume that  $\phi = \neg\phi'$  for some  $\phi' \in \text{CTL}(\text{U}^t, \text{B}^t, \text{B}^l)$ . In that case, we have  $\mathbf{f}_{\text{C-L}(\text{U}^t, \text{B}^t, \text{B}^l)}(\phi) = \neg\mathbf{f}_{\text{C-L}(\text{U}^t, \text{B}^t, \text{B}^l)}(\phi')$ . Hence,  $\mathcal{P}(\phi)$  is a straightforward consequence of  $\mathcal{P}(\phi')$ .
- For all  $\bullet \in \text{U}^t$ , assume that  $\phi = \diamond(\bullet\phi')$  for some  $\diamond \in \{\exists, \forall\}$  and that  $\mathcal{P}(\phi')$  holds. In that case, we have  $\varphi := \mathbf{f}_{\text{C-L}(\text{U}^t, \text{B}^t, \text{B}^l)}(\phi) = \bullet\varphi'$  with  $\varphi' := \mathbf{f}_{\text{C-L}(\text{U}^t, \text{B}^t, \text{B}^l)}(\phi')$ . Let  $i \in \mathbb{N}_1$ . Since  $|\text{Out}^Q(g(i))| = 1$ , it follows that  $K_w^i \models \exists(\bullet\phi')$  iff  $K_w^i \models \forall(\bullet\phi')$  iff:
  - If  $\bullet = \mathbf{X}$ :  $K_w^{i+1} \models \phi'$  iff  $w[i+1:] \models \varphi'$  (by  $\mathcal{P}(\phi')$ ) iff  $w[i:] \models \varphi$ ;
  - If  $\bullet = \mathbf{F}$ : there is some  $j \in \mathbb{N}$ , such that  $K_w^{i+j} \models \phi'$  iff there is some  $j \in \mathbb{N}$ , such that  $w[i+j:] \models \varphi'$  (by  $\mathcal{P}(\phi')$ ) iff  $w[i:] \models \bullet\varphi$ ;
  - If  $\bullet = \mathbf{G}$ : for all  $j \in \mathbb{N}$ , we have  $K_w^{i+j} \models \phi'$  iff for all  $j \in \mathbb{N}$ , we have  $w[i+j:] \models \varphi'$  (by  $\mathcal{P}(\phi')$ ) iff  $w[i:] \models \bullet\varphi$ .

Hence,  $\mathcal{P}(\phi)$  holds.

- The case of binary operators is similar.

In fact,  $\mathcal{P}(\phi)$  holds for all CTL-formulas  $\phi \in \text{CTL}(\text{U}^t, \text{B}^t, \text{B}^l)$ . The lemma follows.  $\square$

We can now define the reduction that we consider.

**Definition 48.** Consider an instance  $\text{In}_{\text{LTL}} = (\text{Prop}, \mathcal{P}, \mathcal{N}, B)$  of the  $\text{LTL}_{\text{Learn}}$  decision problem. We define the input  $\text{In}_{\text{CTL}} = (\text{Prop}, \mathcal{P}', \mathcal{N}', B)$  with:

$$\mathcal{P}' := \{M_w \mid w \in \mathcal{P}\}$$

and

$$\mathcal{N}' := \{M_w \mid w \in \mathcal{N}\}$$

Clearly this reduction can be computed in logarithmic space. Let us now show that it satisfies the desired property.

**Lemma 49.** *Consider a set of propositions  $\text{Prop}$ , sets of operators  $U^t \subseteq \text{Op}_{U^n}$ ,  $B^t \subseteq \text{Op}_{B^n}^{\text{tp}}$  and  $B^l \subseteq \text{Op}_{B^n}^{\text{lg}}$ , and  $k \in \mathbb{N} \cup \{\infty\}$ . Then, the input  $\text{In}_{\text{LTL}}$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(U^t, B^t, B^l, k)$  decision problem if and only if  $\text{In}_{\text{CTL}}$  is a positive instance of the  $\text{CTL}_{\text{Learn}}(U^t, B^t, B^l, k)$  decision problem.*

*Proof.* Assume that  $\text{In}_{\text{LTL}}$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(U^t, B^t, B^l)$  decision problem. Let  $\varphi_{\text{LTL}}$  be an LTL-formula in  $\text{LTL}(U^t, B^t, B^l, k)$  separating  $\mathcal{P}$  and  $\mathcal{N}$  of size at most  $B$ . Consider the CTL-formula  $\varphi_{\text{CTL}} := f_{\text{L-C}(U^t, B^t, B^l)}(\varphi_{\text{LTL}})$ . By Proposition 46, we have  $|\varphi_{\text{CTL}}| = |\varphi_{\text{LTL}}| \leq B$  and  $f_{\text{C-L}(U^t, B^t, B^l)}(\varphi_{\text{CTL}}) = \varphi_{\text{LTL}}$ . Consider now any  $M_w \in \mathcal{P}'$  with  $w \in \mathcal{P}$ . By Lemma 47, we have  $M_w \models \varphi_{\text{CTL}}$  if and only if  $w \models f_{\text{C-L}(U^t, B^t, B^l)}(\varphi_{\text{CTL}}) = \varphi_{\text{LTL}}$ . Since  $w \in \mathcal{P}$ , it follows that  $w \models \varphi_{\text{LTL}}$  and  $M_w \models \varphi_{\text{CTL}}$ . This is similar for any  $M_w \in \mathcal{N}'$  with  $w \in \mathcal{N}$ . Hence,  $\text{In}_{\text{CTL}}$  is a positive instance of the  $\text{CTL}_{\text{Learn}}(U^t, B^t, B^l, k)$  decision problem.

Assume now that  $\text{In}_{\text{CTL}}$  is a positive instance of the  $\text{CTL}_{\text{Learn}}(U^t, B^t, B^l, k)$  decision problem. Let  $\varphi_{\text{CTL}}$  be a CTL-formula separating  $\mathcal{P}'$  and  $\mathcal{N}'$  of size at most  $B$ . Consider the LTL-formula  $\varphi_{\text{LTL}} := f_{\text{C-L}(U^t, B^t, B^l)}(\varphi_{\text{CTL}})$ . By Proposition 46, we have  $|\varphi_{\text{LTL}}| \leq |\varphi_{\text{CTL}}| \leq B$ . Consider now any  $w \in \mathcal{P}$ . By Lemma 47, we have  $M_w \models \varphi_{\text{CTL}}$  if and only if  $w \models \varphi_{\text{LTL}}$ . Since  $M_w \in \mathcal{P}'$ , it follows that  $M_w \models \varphi_{\text{CTL}}$  and  $w \models \varphi_{\text{LTL}}$ . This is similar for any  $w \in \mathcal{N}$ . Hence,  $\text{In}_{\text{LTL}}$  is a positive instance of the  $\text{LTL}_{\text{Learn}}(U^t, B^t, B^l, k)$  decision problem.  $\square$

The proof of Theorem 43 is now direct.

*Proof.* It is straightforward consequence of Lemma 49 and the fact that the reduction from Definition 48 can be computed in logarithmic space.

For all  $k \in \mathbb{N}_1$ , one can straightforwardly simulate a Kripke structure by a concurrent game structure with  $k$  agents by making  $k-1$  agents having only one action. Hence,  $\text{ATL}_{\text{Learn}}^k(U^t, B^t, B^l, n)$  is at least as hard as  $\text{CTL}_{\text{Learn}}(U^t, B^t, B^l, n)$ , under logarithmic space reduction.  $\square$

## 4 Learning formulas using only unary operators

We have seen that, regardless of the operators considered, CTL learning (and therefore ATL learning as well) is at least as hard as LTL learning. Furthermore, we have also seen that LTL learning is NP-hard as soon as any non-unary operator is allowed. In this section, we focus on the learning problems without non-unary operators in order to be able to distinguish the complexity of LTL, CTL and different kinds of ATL learning. In this setting, we show that:

- LTL learning can now be decided in logarithmic space. To establish this result, we use simple results on equivalences of LTL-formulas. Some of these equivalences were already proven in [24].
- CTL learning remains NP-complete, this is proved again via a reduction from the hitting set problem. The reduction relies heavily on the use of the next operator  $\mathbf{X}$ . On the other hand, CTL learning without the next operator  $\mathbf{X}$  is equivalent to the CTL learning with formulas of size at most 5, and is in NL (i.e. it can be decided in non-deterministic logarithmic space).
- On the other hand, ATL learning with at least two agents and at least two operators in  $\{\mathbf{F}, \mathbf{G}, \neg\}$  is still NP-complete. The reduction is an adaptation of the reduction for the CTL case that makes use of the additional players to mimic the behavior of the next operator

$\mathbf{X}$  with both the eventually and globally operators  $\mathbf{F}$  and  $\mathbf{G}$ . This is the most technical reduction of the paper. However, if one only allows the use of either  $\mathbf{F}$  or  $\mathbf{G}$  without negations, then the ATL learning with at most two agents can be decided in polynomial time as, given a bound  $k$  and a set of propositions  $\text{Prop}$ , the number of formulas to check is polynomial in  $k$  and  $|\text{Prop}|$ .

- Finally, ATL learning with at least three agents remains NP-complete even if only of the three operators  $\mathbf{X}, \mathbf{F}, \mathbf{G}$  is allowed. The reduction is an adaptation of the previous one where the third player is used to replace one of the operators  $\mathbf{F}$  or  $\mathbf{G}$ . Hence, in this setting, we do not distinguish, complexity-wise, the ATL learning problems with a fixed number of agents (at least 3) and a number of agents as part of the input. This will be done in the next section.

## 4.1 LTL learning

We first focus on the case of LTL learning. The goal of this subsection is to show the proposition below.

**Proposition 50.** *For all sets  $U^t \subseteq \text{Op}_{U_n}$  and  $B^l \subseteq \text{Op}_{B_{\text{Bin}}}^{\text{lg}}$ , and  $n \in \mathbb{N}$ , the decision problem  $\text{LTL}_{\text{Learn}}(U^t, \emptyset, B^l, n)$  is in L.*

To establish this proposition, we first consider LTL-formulas that do not use any binary operators. First of all, since we consider ultimately periodic words, we have the (well-known, see for instance [24, Proposition 8]) equivalences below.

**Observation 51.** *Consider a non-empty set of propositions  $\text{Prop}$  and some  $k \in \mathbb{N}$ . For all LTL-formulas  $\varphi$  on  $\text{Prop}$ , we have:*

1.  $\mathbf{F} \mathbf{X}^k \varphi \equiv \mathbf{X}^k \mathbf{F} \varphi$
2.  $\mathbf{G} \mathbf{X}^k \varphi \equiv \mathbf{X}^k \mathbf{G} \varphi$
3.  $\mathbf{F} \mathbf{F} \varphi \equiv \mathbf{F} \varphi$
4.  $\mathbf{G} \mathbf{G} \varphi \equiv \mathbf{G} \varphi$
5.  $\mathbf{F} \mathbf{G} \mathbf{F} \varphi \equiv \mathbf{G} \mathbf{F} \varphi$
6.  $\mathbf{G} \mathbf{F} \mathbf{G} \varphi \equiv \mathbf{F} \mathbf{G} \varphi$

*Proof.* Consider an ultimately periodic word  $w = u \cdot v^\omega \in (2^{\text{Prop}})^\omega$ . We prove the first, third and fifth items, the other ones are obtained by duality.

1. We have  $w \models \mathbf{F} \mathbf{X}^k \varphi$  iff there is some  $i \geq 1$  such that  $w[i:] \models \mathbf{X}^k \varphi$  iff there is some  $i \geq 1$  such that  $w[i+k:] \models \varphi$  iff there is some  $j \geq k+1$  such that  $w[j:] \models \varphi$  iff  $w[k+1:] \models \mathbf{F} \varphi$  iff  $w \models \mathbf{X}^k \mathbf{F} \varphi$ .
3. We have  $w \models \mathbf{F} \mathbf{F} \varphi$  iff there is some  $i \geq 1$  such that  $w[i:] \models \mathbf{F} \varphi$  iff there is some  $i \geq 1$  and some  $j \geq i$  such that  $w[i+j:] \models \varphi$  iff there is some  $j \geq 1$  such that  $w[j:] \models \varphi$  iff  $w \models \mathbf{F} \varphi$ .
5. Straightforwardly, we have  $\mathbf{G} \mathbf{F} \varphi \implies \mathbf{F} \mathbf{G} \mathbf{F} \varphi$ . On the other hand, assume that  $w \models \mathbf{F} \mathbf{G} \mathbf{F} \varphi$ , i.e. that there is some  $i \geq 1$  such that  $w[i:] \models \mathbf{G} \mathbf{F} \varphi$ . Thus, for all  $j \geq i$ , we have  $w[j:] \models \mathbf{F} \varphi$ , i.e. there is some  $k_j \geq j$  such that  $w[k_j:] \models \varphi$ . Let us show that  $w \models \mathbf{G} \mathbf{F} \varphi$ . Let  $j \geq 1$  and  $j' := \max(i, j) \geq i$ . We have  $w[k_{j'}:] \models \varphi$ , hence  $w[j:] \models \mathbf{F} \varphi$  since  $k_{j'} \geq j' \geq j$ . Since this holds for all  $j \geq 1$ , we have  $w \models \mathbf{G} \mathbf{F} \varphi$ .

□

In turn, let us consider the definition below of sequences of LTL-operators that we will consider.

**Definition 52.** Consider some  $U^t \subseteq \{\mathbf{X}, \mathbf{F}, \mathbf{G}, \neg\}$ . We let:

$$Qt_{\mathbf{X}}(U^t) := \{Y^k \mid k \in \mathbb{N}, Y \in \{\epsilon\} \cup (U^t \cap \{\mathbf{X}\})\}$$

and

$$Qt_{\mathbf{F}, \mathbf{G}}(U^t) := \begin{cases} \{\mathbf{F}, \mathbf{G}, \mathbf{F}\mathbf{G}, \mathbf{G}\mathbf{F}\} & \text{if } \mathbf{F}, \mathbf{G} \in U^t \\ \{\mathbf{F}\} & \text{if } \mathbf{F} \in U^t, \neg \notin U^t \\ \{\mathbf{F}, \neg\mathbf{F}\neg, \mathbf{F}\neg\mathbf{F}, \neg\mathbf{F}\neg\mathbf{F}\} & \text{if } \mathbf{F}, \neg \in U^t \\ \{\mathbf{G}\} & \text{if } \mathbf{G} \in U^t, \neg \notin U^t \\ \{\mathbf{G}, \neg\mathbf{G}\neg, \neg\mathbf{G}\neg\mathbf{G}, \mathbf{G}\neg\mathbf{G}\neg\} & \text{if } \mathbf{G}, \neg \in U^t \\ \{\epsilon\} & \text{if } \mathbf{F}, \mathbf{G} \in U^t \end{cases}$$

and

$$Qt_{\neg}(U^t) := \{\epsilon\} \cup U^t \cap \{\neg\}$$

Then, we let:

$$\text{SeqQt}_{\text{LTL}}(U^t) := \{Q_{\mathbf{X}} \cdot Q_{\mathbf{F}, \mathbf{G}} \cdot Q_{\neg} \mid Q_{\mathbf{X}} \in Qt_{\mathbf{X}}(U^t), Q_{\mathbf{F}, \mathbf{G}} \in Qt_{\mathbf{F}, \mathbf{G}}(U^t), Q_{\neg} \in Qt_{\neg}(U^t)\}$$

We deduce the corollary below.

**Corollary 53.** Consider a non-empty set of propositions  $\text{Prop}$ , and some  $U^t \subseteq \{\mathbf{X}, \mathbf{F}, \mathbf{G}, \neg\}$ . For any LTL-formula  $\varphi = \text{Qt} \cdot \varphi' \in \text{LTL}(\text{Prop}, U^t, \emptyset, \emptyset, 0)$ , where  $\text{Qt}$  is a sequence of operators and  $\varphi' \in \text{LTL}(\text{Prop}, U^t, \emptyset, \emptyset, 0)$  is an LTL-formula, there is a sequence of operators  $\text{Qt}' \in \text{SeqQt}_{\text{LTL}}(U^t)$  such that, for  $\varphi'' := \text{Qt}' \cdot \varphi' \in \text{LTL}(\text{Prop}, U^t, \emptyset, \emptyset, 0)$ , we have  $\varphi \equiv \varphi''$  and  $|\varphi''| \leq |\varphi|$ .

*Proof.* This is a direct consequence of Observation 51 and the fact that, for all LTL-formulas  $\varphi$ , we have  $\varphi \equiv \neg\neg\varphi$ ,  $\mathbf{F}\varphi \equiv \neg\mathbf{G}\neg\varphi$  and  $\mathbf{G}\varphi \equiv \neg\mathbf{F}\neg\varphi$ . □

Let us now consider LTL-formulas with (a bounded amount of occurrences of) binary operators.

**Definition 54.** Consider a non-empty set of propositions  $\text{Prop}$ , and some  $U^t \subseteq \text{Op}_{U_n}$  and  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$ . We define inductively the set  $\text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, \infty) \subseteq \text{LTL}(\text{Prop}, U^t, \emptyset, B^l, \infty)$ :

- $p \in \text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, \infty)$ , for all  $p \in \text{Prop}$ ;
- $\text{Qt} \cdot \varphi \in \text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, \infty)$ , for all  $\text{Qt} \in \text{SeqQt}_{\text{LTL}}(U^t)$  and  $\varphi \in \text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, \infty)$ ;
- $\varphi_1 \bullet \varphi_2 \in \text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, \infty)$ , for all  $\bullet \in B^l$  and  $\varphi_1, \varphi_2 \in \text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, \infty)$ .

Then, for all  $n \in \mathbb{N}$ , the set of LTL-formulas  $\text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, n)$  is defined by  $\text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, n) := \text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, \infty) \cap \text{LTL}(\text{Prop}, U^t, \emptyset, B^l, n)$ .

To conclude and prove Proposition 50, we establish the three following facts, for sets  $U^t \subseteq \text{Op}_{U_n}$  and  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$ : 1) for all instances  $I = (\text{Prop}, \mathcal{P}, \mathcal{N}, k)$  of the decision problem  $\text{LTL}_{\text{Learn}}(U^t, \emptyset, B^l, n)$ ,  $I$  is a positive instance if and only if there is an  $\text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, n)$ -formula of size at most  $k$  accepting  $\mathcal{P}$  and rejecting  $\mathcal{N}$ . Furthermore, let us fix a bound  $n \in \mathbb{N}$  on the number of occurrences of binary operators, then: 2) the number of  $\text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, n)$ -formulas of size at most  $k$  is polynomial in  $k$  and  $|\text{Prop}|$ ; and 3) there is a logarithmic-space algorithm that decides if an ultimately-periodic word satisfies  $\text{LTL}_{\text{Unif}}(\text{Prop}, U^t, \emptyset, B^l, n)$ -formulas.

Before we argue that these facts hold, let us show that they directly imply Proposition 50.

*Proof.* Consider the decision problem  $\text{LTL}_{\text{Learn}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$ . With fact 1, deciding if an instance  $(\text{Prop}, \mathcal{P}, \mathcal{N}, k)$  is positive amounts to deciding the existence of an  $\text{LTL}_{\text{Unif}}(\text{Prop}, \mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$ -formula of size at most  $k$  accepting  $\mathcal{P}$  and rejecting  $\mathcal{N}$ . With facts 2) and 3), we can design a logarithmic space algorithmic solving that problem: it suffices to have a counter with which we enumerate all the polynomially-many  $\text{LTL}_{\text{Unif}}(\text{Prop}, \mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$ -formulas to consider, and then check if one them does accept  $\mathcal{P}$  and reject  $\mathcal{N}$ .  $\square$

Let us now argue that these three facts hold. The first two facts are rather straightforward. Indeed, Fact 1) is direct consequence of Corollary 53. In addition, Fact 2) is a direct consequence (which can be proved straightforwardly by induction on  $n$ ) of the fact that the number of sequences of operators in  $\text{SeqQt}_{\text{LTL}}(\mathbf{U}^t)$  of size at most  $k$  is polynomial (in fact, linear) in  $k$ . As it is more involved, we state Fact 3 in a lemma below. Its proof concludes this subsection.

**Lemma 55.** *Consider a bound  $n \in \mathbb{N}$ . The following decision problem can be decided in logarithmic space:*

- *Input:* an  $\text{LTL}_{\text{Unif}}(\text{Prop}, \mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$ -formula  $\varphi$ , and an ultimately periodic word  $w = u \cdot v^\omega$ ;
- *Output:* Yes iff  $w \models \varphi$ .

*Proof.* The recursive algorithm depicted in Figure 1 (in which  $\bullet$  refers to a binary operator) straightforwardly solves the decision problem (by simply following the LTL semantics). We have to argue that it can be implemented in logarithmic space. To execute the algorithmic, it is sufficient to keep a pointer to the current position in the word, plus additional pointers:

- To keep track of the sub-formulas currently being evaluated, and the intermediary results already computed, which is necessary with binary operators. Since there are at most  $n$  occurrences of binary operators, the total number of intermediary results to keep track of is bounded by  $2^n$ .
- To keep track of the indices being evaluated, which is necessary with the  $\mathbf{F}$  and  $\mathbf{G}$  operators. However, since we consider  $\text{LTL}_{\text{Unif}}(\text{Prop}, \mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$ -formulas, between binary operators, there is at most two  $\mathbf{F}$  operators and  $\mathbf{G}$  operators. Thus, again, the amount of pointers sufficient is bounded by  $2n$ .

Overall, the total number of pointers sufficient to keep track of everything is bounded, independently of the input. Thus, Algorithm  $\text{DecideLTL}_{\text{Unif}}$  can be implemented in logarithmic space.  $\square$

## 4.2 Abstract recipe for NP-hardness proofs and binary operators

As can be seen in Table 1, we are going to establish three NP-hardness results in the following cases: CTL-learning with the operator  $\mathbf{X}$ , ATL-learning with two agents and both operators  $\mathbf{F}$  and  $\mathbf{G}$ , and ATL-learning with three agents and the operator  $\mathbf{F}$ , or the operator  $\mathbf{G}$ . Although these reductions differ, they all share a similar structure. The goal of this subsection is: first, to give the abstract recipe that we follow for all these reductions; and second, to state and prove the relevant lemmas to handle binary operators in our reductions.



```

 $n \leftarrow |w|, m \leftarrow u$ 
if  $\varphi = p$  then
  | Return  $p \in w[i]$ ;
else if  $\varphi = X \varphi'$  then
  | if  $i < n$  then
  | | DecideLTLUnif( $\varphi', w, i + 1$ );
  | else
  | | DecideLTLUnif( $\varphi', w, m + 1$ )
  | end
end
else if  $\varphi = \neg \varphi'$  then
  | not DecideLTLUnif( $\varphi', w, i$ )
end
else if  $\varphi = F \varphi'$  then
  | Test  $\leftarrow$  False
  | for  $j \in [\min(i, m + 1), n]$  do
  | | if DecideLTLUnif( $\varphi', w, j$ ) then
  | | | Test  $\leftarrow$  True
  | | Return Test
  | end
end
else if  $\varphi = G \varphi'$  then
  | Test  $\leftarrow$  True
  | for  $j \in [\min(i, m + 1), n]$  do
  | | if notDecideLTLUnif( $\varphi', w, j$ ) then
  | | | Test  $\leftarrow$  False
  | | Return Test
  | end
end
else if  $\varphi = \varphi_1 \bullet \varphi_2$  then
  | Res1  $\leftarrow$  DecideLTLUnif( $\varphi_1, w, i$ )
  | Res2  $\leftarrow$  DecideLTLUnif( $\varphi_2, w, i$ )
  | Return Res1  $\bullet$  Res2
end

```

Figure 1: Algorithm DecideLTL<sub>Unif</sub>( $\varphi, w, i$ ).

### 4.2.1 Abstract recipe

Since the formulas to learn can only contain a bounded amount of binary operators, we cannot define a hitting set from a (small enough) separating formula by looking at the propositions that it uses, as we have done for LTL in Section 3. Instead, our reduction focuses on formulas using only unary operators (and therefore a single proposition). In that context, we create a sample (and a bound  $B$ ) such that all separating (unary) formulas have a specific shape, and there is bijection between subsets  $H \subseteq [1, \dots, l]$  and formulas  $\phi(l, H)$  of that specific shape. This correspondence allows us to extract a hitting set. Note that, however, although the formulas that we consider can only use a bounded amount of binary operators, they can still use some binary operators. Therefore, to be able to only focus on unary operators, we first need to control how the (separating) formulas use binary operators. More specifically, we follow the abstract recipe below, from an  $(l, C, k)$  of the hitting set problem:

- a) We first handle binary operators. That is, given any  $n \in \mathbb{N}$  and some binary operators  $\bullet_2 \in \mathbf{B}^l$ , we consider  $n$  propositions  $p_1, \dots, p_n$  and several positive  $\mathcal{P}_{n, \bullet}$  and negative  $\mathcal{N}_{n, \bullet}$  structures such that a separating formula has to use binary operators to express properties with the  $n$  propositions  $p_1, \dots, p_n$ .
- b) We can now focus on unary formulas. We define the bound  $B$  and additional positive  $\mathcal{P}_{\text{Un}}$  and negative  $\mathcal{N}_{\text{Un}}$  structures (using only the proposition  $p$ ) that “eliminate” certain (unary) operators or pattern of (unary) operators. This way we ensure that any unary formula separating  $\mathcal{P}_{\text{Un}}$  and  $\mathcal{N}_{\text{Un}}$  will be of the form  $\phi(l, H)$ , for some  $H \subseteq [1, \dots, l]$ .
- c) We can finally encode the hitting set problem itself. We define a negative structure (on  $p$ ) that a unary formula  $\phi(l, H)$  accepts if and only if  $|H| \geq k + 1$ .
- d) For all  $1 \leq i \leq n$ , we define a positive structure (on  $p$ ) that a unary formula  $\phi(l, H)$  accepts if and only if  $H \cap C_i \neq \emptyset$ .

By construction, the instance of the learning decision problem that we obtain is a positive instance if and only if  $(l, C, k)$  also is.

### 4.2.2 Handling binary operators

The statement of this section deal with ATL-structures and formulas because they will be used for the next three NP-hardness proof.

**Notations and definitions** First of all, we will use notations akin to that of regular languages to describe the formulas that we will consider.

**Notation 56.** For any ATL-formula  $\phi$  and set of operators  $O$ , we denote by  $O^* \phi$  the set of ATL-formulas beginning with finitely many operators in  $O$  followed by the ATL-formula  $\phi$ . Furthermore, when  $O$  is not a singleton, its elements may be enumerated with commas.

For the ATL-reductions, we will use turn-based game structures, where, at each state, only one agent is choosing the next state. Note that Kripke structure can be seen as turn-based structures, with only one player.

**Definition 57.** Given any coalition of agents  $A$ , an  $A$ -turn-based game structure  $T$  is defined by a tuple  $T = \langle Q, I, \text{Prop}, \pi, \text{AgSt}, \text{Succ} \rangle$  where  $\text{AgSt} : Q \rightarrow A$  maps every state to an agent in  $A$  and  $\text{Succ} : Q \rightarrow 2^Q$  maps every state  $q \in Q$  its set of successor states where the agent  $\text{AgSt}(q)$  can choose to go. Note that when a state has only one successor, i.e. one outgoing edge, the

identity of the agent owning the state is irrelevant. The coalitions  $A$  of agents that we consider are always such that  $A \subseteq \{1, 2, 3\}$ .

Interestingly for us, when evaluated on turn-based structures, ATL-formulas satisfy the classical equivalences w.r.t. negations. Let us introduce below a notation that refers to the dual of the operators considered.

**Definition 58.** For all  $k \in \mathbb{N}_1$ ,  $A \subseteq [1, \dots, k]$ , and  $H \in \{\mathbf{X}, \mathbf{F}, \mathbf{G}\}$ , we let:

$$\overline{\langle\langle A \rangle\rangle H}^k := \langle\langle [1, \dots, k] \setminus A \rangle\rangle \bar{H}$$

where  $\bar{\mathbf{X}} := \mathbf{X}$ ,  $\bar{\mathbf{F}} := \mathbf{G}$  and  $\bar{\mathbf{G}} := \mathbf{F}$ .

This definition satisfies the proposition below.

**Proposition 59.** Let  $k \in \mathbb{N}_1$  and  $\text{Ag} := [1, \dots, k]$ . For all  $A \subseteq \text{Ag}$ ,  $H \in \{\mathbf{X}, \mathbf{F}, \mathbf{G}\}$  and any ATL-formula  $\phi$ , when evaluated on  $\text{Ag}$ -turn-based game structures, for all coalitions of agents  $A \subseteq [1, \dots, k]$ , we have the equivalence  $\neg \langle\langle A \rangle\rangle H \phi \equiv \overline{\langle\langle A \rangle\rangle H}^k \neg \phi$ .

*Proof.* This is a consequence of the fact that two-player (with one player representing the coalition  $A$  and the other the coalition  $\text{Ag} \setminus A$ ) turn-based reachability/safety games (respectively for the  $\mathbf{F}/\mathbf{G}$  operator) are determined: from any starting state, either one of the players has winning (positional) strategies.  $\square$

We can use this property to define a way to remove negations from sequences of unary operators, while maintaining the semantics of the formula (and not increasing the size). This is done in the definition.

**Definition 60.** Let  $k \in \mathbb{N}_1$ , and  $\text{Ag} := [1, \dots, k]$ . For  $\text{U}^t \subseteq \text{Op}_{\text{Un}}$ , we let  $\text{Op}(k, \text{U}^t) := \{\langle\langle A \rangle\rangle H \mid A \subseteq \text{Ag}, H \in \text{U}^t\} \cup \{\neg\} \cap \text{U}^t$ . We define inductively on  $(\text{Op}(k, \text{U}^t))^+$  a function  $\text{UnNeg} : (\text{Op}(k, \text{U}^t))^* \times \{0, 1\} \rightarrow (\text{Op}(k, \{\mathbf{X}, \mathbf{F}, \mathbf{G}, \neg\}))^* \times \{0, 1\}$  as follows:

- For all  $x \in \{0, 1\}$ , we let  $\text{UnNeg}(\epsilon, x) := (\epsilon, x)$ .
- For all  $O \in \text{Op}(k, \text{U}^t)$ ,  $\text{Qt} \in (\text{Op}(k, \text{U}^t))^*$ , and  $x \in \{0, 1\}$ , we let:

$$\text{UnNeg}(O \cdot \text{Qt}, x) := \begin{cases} (O \cdot \text{Qt}', x') & \text{if } x = 0 \\ (\bar{O}^k \cdot \text{Qt}', x') & \text{if } x = 1 \end{cases}$$

for  $(\text{Qt}', x') := \text{UnNeg}(\text{Qt}, x)$ .

- For all  $x \in \{0, 1\}$  and  $\text{Qt} \in (\text{Op}(k, \text{U}^t))^*$ , we let  $\text{UnNeg}(\neg \text{Qt}, x) := \text{UnNeg}(\text{Qt}, 1 - x)$ .

This definition satisfies the lemma below.

**Lemma 61.** Let  $k \in \mathbb{N}_1$ ,  $\text{Ag} := [1, \dots, k]$ ,  $\text{U}^t \subseteq \text{Op}_{\text{Un}}$ , and  $\text{Qt} \in (\text{Op}(k, \text{U}^t))^*$ . We have:

- For all  $x \in \{0, 1\}$ , letting  $(\text{Qt}, y) := \text{UnNeg}(\text{Qt}, x)$ , we have  $\text{Qt}' \in (\text{Op}(k, \{\mathbf{X}, \mathbf{F}, \mathbf{G}\}))^*$  and if  $\mathbf{F} \in \text{U}^t$  if and only if  $\mathbf{G} \in \text{U}^t$ , then  $\text{Qt}' \in (\text{Op}(k, \text{U}^t \setminus \{\neg\}))^*$ ;
- For all  $x \in \{0, 1\}$ ,  $(\text{Qt}, y) := \text{UnNeg}(\text{Qt}, x)$ , we have  $|\text{Qt}'| \leq |\text{Qt}|$ ;
- Consider an ATL-formula  $\phi$ . For all  $i \in \{0, 1\}$ , we let  $(\text{Qt}_i, x_i) := \text{UnNeg}(\text{Qt}, i)$  and  $\psi_i := \phi$  if  $x_i = 0$ , and  $\psi_i := \neg \phi$  otherwise. Then, we have  $\text{Qt} \cdot \phi \equiv \text{Qt}_0 \cdot \psi_0$  and  $\neg \text{Qt} \cdot \phi \equiv \text{Qt}_1 \cdot \psi_1$ .

*Proof.* The first item follows from Definition 58. The second item follows from Definition 60. The third item can be obtained by induction on  $\mathbf{Qt}$ , by successively applying Proposition 59.  $\square$

Now, let us define the kind of turn-based game structures that we will consider. Note that there is a slight difference with the informal explanations of Section 4.2.1: in addition to the  $n$  propositions  $p_1, \dots, p_n$ , we do not consider a single proposition  $p$  but rather two propositions  $p$  and  $\bar{p}$ . This is done to negate formulas without using negations (i.e. informally,  $\bar{p}$  will be equivalent to  $\neg p$  on the structures that we will consider).

**Definition 62.** Let  $n \in \mathbb{N}$ . We let  $\mathbf{Prop}_n := \{p, \bar{p}\} \cup \mathbf{Prop}'_n$  where  $\mathbf{Prop}'_n := \{p_1, \dots, p_n\}$  be two sets of propositions. Then, for all  $S \subseteq \mathbf{Prop}'_n$ , a turn-based structure  $T$  is  $(n, S)$ -proper if, for all states  $q$  in  $T$ , we have  $\pi(q) \in \{S \cup \{p\}, S \cup \{\bar{p}\}\}$ .

Let  $k \in \mathbb{N}_1$  and  $\mathbf{Ag} := [1, \dots, k]$ . Given any two ATL-formulas  $\phi, \phi'$ , we denote by  $\phi \equiv_{k,n,S} \phi'$  the fact that  $\phi$  and  $\phi'$  are equivalent on  $(n, S)$ -proper  $\mathbf{Ag}$ -turn-based structures, i.e. for all proper turn-based structures  $T$ , we have  $T \models \phi$  if and only if  $T \models \phi'$ .

A structure  $T$  is trivial if it contains a single self-looping state. It can be seen as a turn-based structure with any number of agents. In addition, such a structure is entirely defined by the set  $S$  of propositions labeling the unique state of the structure. It is denoted  $T(S)$ .

We can now state the two lemmas that will let us properly handle binary operators in our NP-hardness proofs. The first one, Lemma 63, states that from an ATL-formula  $\phi$  using  $k$  binary operators and featuring  $k$  propositions in  $\mathbf{Prop}'_n$  (for some  $k \leq n$ ), for all  $S \subseteq \mathbf{Prop}'_n$ , we can extract a unary ATL-formula  $\phi'$  equivalent to  $\phi$  on  $S$ -proper structures. This is stated below.

**Lemma 63.** Let  $k \in \mathbb{N}_1$ . Consider a set of unary operators  $\mathbf{U}^t \in \mathbf{Op}_{\mathbf{U}^t}$  such that  $\mathbf{G} \in \mathbf{U}^t$  if and only if  $\mathbf{F} \in \mathbf{U}^t$ , and some set of binary logical operator  $\mathbf{B}^l \subseteq \mathbf{Op}_{\mathbf{B}^l}^{\text{lg}}$ . Let  $n \in \mathbb{N}$  and  $i \leq n$ . For all ATL-formulas  $\phi \in \text{ATL}^k(\mathbf{Prop}, \mathbf{U}^t, \emptyset, \mathbf{B}^l, i)$  such that  $|\mathbf{Prop}(\phi) \cap \mathbf{Prop}'_n| = i$  and for all  $S \subseteq \mathbf{Prop}'_n$ , there are two ATL-formulas  $\psi, \hat{\psi} \in \text{ATL}^k(\{p, \bar{p}\}, \mathbf{U}^t \setminus \{\neg\}, \emptyset, \mathbf{B}^l, 0)$  such that:

- $|\psi|, |\hat{\psi}| \leq |\phi| - 2i$ ; and
- $\phi \equiv_{k,n,S} \psi$  and  $\neg\phi \equiv_{k,n,S} \hat{\psi}$ .

The second lemma, Lemma 64, shows the existence, for all  $n \in \mathbb{N}$  and binary operators  $\bullet \in \mathbf{B}^l$ , of the two sets of structures  $\mathcal{P}_{n,\bullet}$  and  $\mathcal{N}_{n,\bullet}$  mentioned in Step a) in the abstract recipe described in Section 4.2.1. It is stated below.

**Lemma 64.** Let  $k \in \mathbb{N}_1$ . Consider a binary operator  $\bullet \in \mathbf{Op}_{\mathbf{B}^l}^{\text{lg}}$  and some  $n \in \mathbb{N}_1$ . There is some  $S_{n,\bullet} \subseteq \mathbf{Prop}'_n$  and two sets trivial structures  $\mathcal{A}_{n,\bullet}$  and  $\mathcal{B}_{n,\bullet}$  such that:

- If an ATL-formula  $\phi$  distinguishes  $\mathcal{A}_{n,\bullet}$  and  $\mathcal{B}_{n,\bullet}$ , then  $\mathbf{Prop}'_n \subseteq \mathbf{Prop}(\phi)$ .
- There is an ATL-formula  $\phi_{n,\bullet} \in \text{ATL}^k(\mathbf{Prop}^n, \emptyset, \emptyset, \{\bullet\}, n)$  of size  $2n - 1$  and such that, for all positive and negative sets  $\mathcal{P}, \mathcal{N}$  of  $(n, S_{n,\bullet})$ -proper structures, there is  $(\mathcal{P}', \mathcal{N}') \in \{(\mathcal{P} \cup \mathcal{A}_{n,\bullet}, \mathcal{N} \cup \mathcal{B}_{n,\bullet}), (\mathcal{N} \cup \mathcal{B}_{n,\bullet}, \mathcal{P} \cup \mathcal{A}_{n,\bullet})\}$  such that, for all ATL-formulas  $\phi \in \text{ATL}^k(\mathbf{Prop}^n, \mathbf{U}^t \setminus \{\neg\}, \emptyset, \mathbf{B}^l, 0)$ , we have:

$\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$  if and only if  $\psi := \phi \bullet \phi_{n,\bullet}$  accepts  $\mathcal{P}'$  and rejects  $\mathcal{N}'$

With both of those lemmas, we are able to show the theorem below that allows to prove the NP-hardness of the learning decision problems that we consider, with an arbitrary bound on the number of occurrences of binary operators, by showing the NP-hardness of the learning problem without binary operator.

**Theorem 65.** Let  $k \in \mathbb{N}_1$ ,  $\text{Ag} := [1, \dots, k]$ , and  $\text{U}^t \in \text{Op}_{\text{Un}}$ . Assume that there is a function  $f_k$  computable in logarithmic space that takes an input an instance  $(l, C, k')$  of the hitting set problem Hit and returns an instance of the learning decision problem  $\text{ATL}_{\text{Learn}}^k(\text{Un}, \emptyset, \emptyset, 0)$  such that for all instances  $(l, C, k')$ :

- The set of propositions in  $f_k((l, C, k'))$  is  $\text{Prop}_0$ ;
- All structures in  $f_k((l, C, k'))$  are  $(0, \emptyset)$ -proper Ag-turn-based structures;
- The three statements below are equivalent:
  - $(l, C, k')$  is a positive instance of Hit;
  - $f_k((l, C, k'))$  is a positive instance of  $\text{ATL}_{\text{Learn}}^k(\text{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$ ;
  - $f_k((l, C, k'))$  is a positive instance of  $\text{ATL}_{\text{Learn}}^k(\text{U}^t \cup \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ .

Then, for all  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ , the decision problem  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \mathbf{B}^l, n)$  is NP-complete.

*Proof.* Let  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ . As mentioned in Proposition 6, the decision problem  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \mathbf{B}^l, n)$  is in NP. Let us now show that it is NP-hard.

If  $n = 0$  or  $\mathbf{B}^l = \emptyset$ , the two problems  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \emptyset, 0)$  and  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \mathbf{B}^l, n)$  are the same. Furthermore, for all instances  $(l, C, k')$  of Hit, we have that if  $f_k((l, C, k'))$  is a positive instance of  $\text{ATL}_{\text{Learn}}^k(\text{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$ , then it is also a positive instance of  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \emptyset, 0)$ , and similarly if  $f_k((l, C, k'))$  is a positive instance of  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \emptyset, 0)$ , then it is also a positive instance of  $\text{ATL}_{\text{Learn}}^k(\text{U}^t \cup \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ . Therefore,  $f_k((l, C, k'))$  is a positive instance of  $\text{ATL}_{\text{Learn}}^k(\text{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$  if and only if it is a positive instance of  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \emptyset, 0)$ . We can conclude that the decision problem  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \mathbf{B}^l, n)$  is NP-hard.

Assume now that we have  $n \in \mathbb{N}_1$  and  $\mathbf{B}^l \neq \emptyset$ . We let  $\bullet \in \mathbf{B}^l$  and we consider the set of propositions  $S_{n, \bullet} \subseteq \text{Prop}'_n$  and the two sets of trivial structures  $\mathcal{A}_{n, \bullet}$  and  $\mathcal{B}_{n, \bullet}$  from Lemma 64, along with the formula  $\phi_{n, \bullet} \in \text{ATL}^k(\text{Prop}^n, \emptyset, \emptyset, \{\bullet\}, n)$  of size at most  $2n - 1$ . Now, consider an instance  $(l, C, k')$  of the hitting set problem Hit. Let  $\text{In}_0 := f_k((l, C, k')) = (\text{Prop}_0, \mathcal{P}, \mathcal{N}, B)$ . Note that all structures in  $\mathcal{P} \cup \mathcal{N}$  are  $(0, \emptyset)$ -proper Ag-turn-based structures. We let  $\widehat{\mathcal{P}}, \widehat{\mathcal{N}}$  be two sets of Ag-turn-based structures equal to  $\mathcal{P}, \mathcal{N}$  respectively except that the labels of all states is changed from  $x \in \text{Prop}_0 = \{p, \bar{p}\}$  to  $S_{n, \bullet} \cup \{x\} \subseteq \text{Prop}_n$ . That way, all the turn-based structures in  $\widehat{\mathcal{P}}$  and  $\widehat{\mathcal{N}}$  are  $(n, S_{n, \bullet})$ -proper structures. Then, we let  $(\mathcal{P}', \mathcal{N}') \in \{(\widehat{\mathcal{P}} \cup \mathcal{A}_{n, \bullet}, \widehat{\mathcal{N}} \cup \mathcal{B}_{n, \bullet}), (\widehat{\mathcal{N}} \cup \mathcal{B}_{n, \bullet}, \widehat{\mathcal{P}} \cup \mathcal{A}_{n, \bullet})\}$  be as in the second point of Lemma 64 and we define the input  $\text{In}_n := (\text{Prop}_n, \mathcal{P}', \mathcal{N}', B + 2n)$  of the decision problem  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \mathbf{B}^l, n)$ . Note that, the structures in  $\mathcal{A}_{n, \bullet}$  and  $\mathcal{B}_{n, \bullet}$  are defined independently of the input  $(l, C, k')$  and, by assumption, the input  $\text{In}_0$  can be computed in logarithmic space from the instance  $(l, C, k')$ . Therefore, the input  $\text{In}_n$  can also be computed in logarithmic space from the instance  $(l, C, k')$ .

Let us show that  $\text{In}_n$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \mathbf{B}^l, n)$  if and only if  $\text{In}_0$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^k(\text{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$ .

Assume that  $\text{In}_n$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^k(\text{U}^t, \emptyset, \mathbf{B}^l, n)$ . Consider a formula  $\phi \in \text{ATL}_{\text{Learn}}^k(\text{Prop}_n, \text{U}^t, \emptyset, \mathbf{B}^l, n) \subseteq \text{ATL}_{\text{Learn}}^k(\text{Prop}_n, \text{U}^t \cup \{\mathbf{F}, \mathbf{G}\}, \emptyset, \mathbf{B}^l, n)$  of size at most  $B + 2n$  that accepts  $\mathcal{P}'$  and rejects  $\mathcal{N}'$ . In that case, the formula  $\phi$  distinguishes  $\mathcal{A}_{n, \bullet}$  and  $\mathcal{B}_{n, \bullet}$ , and therefore, by Lemma 64, we have  $\text{Prop}'_n \subseteq \text{Prop}(\phi)$ . Hence, by Lemma 63, there are two formulas  $\psi, \widehat{\psi} \in \text{ATL}_{\text{Learn}}^k(\text{Prop}_0, (\text{U}^t \cup \{\mathbf{F}, \mathbf{G}\}) \setminus \{\neg\}, \emptyset, \emptyset, 0)$  such that  $|\psi|, |\widehat{\psi}| \leq |\phi| - 2n \leq B$  and  $\phi \equiv_{k, n, S_{n, \bullet}} \psi$  and  $\neg\phi \equiv_{k, n, S_{n, \bullet}} \widehat{\psi}$ . Let  $\phi' \in \{\psi, \widehat{\psi}\}$  be such that  $\phi' = \psi$  if and only if  $(\mathcal{P}', \mathcal{N}') = (\widehat{\mathcal{P}} \cup \mathcal{A}_{n, \bullet}, \widehat{\mathcal{N}} \cup \mathcal{B}_{n, \bullet})$ . That way, we have that  $\phi' \in \text{ATL}_{\text{Learn}}^k(\text{Prop}_n, \text{U}^t \cup \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$  accepts the set  $\widehat{\mathcal{P}}$  and rejects the set  $\widehat{\mathcal{N}}$ . Since  $\text{Prop}(\phi') \subseteq \text{Prop}_0 = \{p, \bar{p}\}$ , and by definition of  $\widehat{\mathcal{P}}$  and  $\widehat{\mathcal{N}}$ , it follows that  $\phi'$  also accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . Thus,  $\text{In}_0$  is a positive instance of the

decision problem  $\text{ATL}_{\text{Learn}}^k(\mathbf{U}^t \cup \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ , and is therefore also a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^k(\mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$ .

Assume now that  $\text{In}_0$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^k(\mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$ . Consider an ATL-formula  $\phi \in \text{ATL}_{\text{Learn}}^k(\text{Prop}_0, \mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$  of size at most  $B$  that accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . We consider the ATL formula  $\phi' := \phi \bullet \phi_{n,\bullet} \in \text{ATL}_{\text{Learn}}^k(\text{Prop}_0, \mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$ . We have  $|\phi'| \leq |\phi| + |\phi_{n,\bullet}| + 1 \leq B + 2n - 1 + 1 = B + 2n$ . Furthermore, by definition of the sets  $\mathcal{P}', \mathcal{N}'$ , we have that  $\phi'$  accepts  $\mathcal{P}'$  and rejects  $\mathcal{N}'$  since  $\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ <sup>4</sup>. Thus,  $\text{In}_n$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^k(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$ .

Overall, we obtain that  $\text{In}_n$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^k(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  if and only if  $(l, C, k')$  is a positive instance of the hitting set problem Hit. Hence, the decision problem  $\text{ATL}_{\text{Learn}}^k(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  is NP-hard.  $\square$

Let us now prove these two Lemmas 63 and 64. We start with the proof of Lemma 63, but before we proceed to it, we state and prove below a crucial lemma regarding the shape of formulas that use a bounded amount of binary operators and feature many propositions.

**Lemma 66.** *For all  $k \in \mathbb{N}$ , for all ATL-formulas  $\phi$ , if  $|\phi|_{\text{Bin}} = k$ , then  $|\text{Prop}(\phi)| \leq k + 1$ . In addition, if  $|\text{Prop}(\phi)| = k + 1$ , and  $\phi = \phi_1 \bullet \phi_2$  for some binary operator  $\bullet \in \mathbf{B}^l$ , and ATL-formulas  $\phi_1, \phi_2$ , then  $\text{Prop}(\phi_1) \cap \text{Prop}(\phi_2) = \emptyset$ .*

*Proof.* We prove this lemma by induction on  $k$ . It straightforwardly holds for  $k = 0$ . Assume now that it holds for all  $i \leq k$ , for some  $k \in \mathbb{N}$ . Then, consider an ATL-formula  $\phi$  for which  $|\phi|_{\text{Bin}} = k + 1$ . Let us first assume that  $\phi = \phi_1 \bullet \phi_2$ , for some with binary operator  $\bullet$ , and ATL-formulas  $\phi_1, \phi_2$ . We are going to modify the formula  $\phi_1$ . To do so, we consider a fresh proposition  $x \notin \text{Prop}(\phi)$ . Then, for all ATL-formulas  $\psi, \psi'$ , we let  $\text{ch}_x(\psi, \psi') := x$  if  $\psi \in \text{SubF}(\phi_2)$ , and  $\text{ch}_x(\psi, \psi') := \psi'$  otherwise. Now, we define inductively a function  $\text{tr}_x$  on ATL-formulas as follows:

- for all propositions  $p$ , we let  $\text{tr}_x(p) := \text{ch}_x(p, x)$ ;
- for all unary operators  $\bullet_1$  and ATL-formulas  $\psi$ , we let  $\text{tr}_x(\bullet_1 \psi) := \bullet_1 \text{ch}_x(\psi, \text{tr}_x(\psi))$ ;
- for all binary operators  $\bullet_2$ , and  $\psi, \psi'$  two ATL-formulas, we let  $\text{tr}_x(\psi \bullet_2 \psi') := \text{ch}_x(\psi, \text{tr}_x(\psi)) \bullet_2 \text{ch}_x(\psi', \text{tr}_x(\psi'))$ .

We are going to apply the function  $\text{tr}_x$  to the formula  $\phi_1$ , and then use the properties satisfied by the obtained formula. Thus, we show by induction on ATL-formulas  $\psi$  the property  $\mathcal{P}(\psi)$ :  $\text{SubF}(\text{ch}_x(\psi, \text{tr}_x(\psi))) \cap \text{SubF}(\phi_2) = \emptyset$  and  $\text{SubBin}(\text{ch}_x(\psi, \text{tr}_x(\psi))) \subseteq \text{tr}_x(\text{SubBin}(\psi) \setminus \text{SubBin}(\phi_2))$ . First of all, for all ATL-formulas, if  $\psi \in \text{SubF}(\phi_2)$ , then  $\text{ch}_x(\psi, \text{tr}_x(\psi)) = x$ , in which case  $\text{SubF}(\text{ch}_x(\psi, \text{tr}_x(\psi))) \cap \text{SubF}(\phi_2) = \emptyset$ , and  $\text{SubBin}(\text{ch}_x(\psi, \text{tr}_x(\psi))) = \emptyset$ . Hence,  $\mathcal{P}(\psi)$  always holds in that case. Thus, we will focus on the cases where  $\psi \notin \text{SubF}(\phi_2)$  and  $\text{ch}_x(\psi, \text{tr}_x(\psi)) = \text{tr}_x(\psi)$ .

Furthermore,  $\mathcal{P}(p)$  straightforwardly holds for all propositions  $p$ . Assume now that  $\mathcal{P}(\psi)$  holds for some ATL-formula  $\psi$ . Consider any unary operator  $\bullet_1$ . We have  $\text{tr}_x(\bullet_1 \psi) = \bullet_1 \text{ch}_x(\psi, \text{tr}_x(\psi))$ , hence  $\text{SubF}(\text{tr}_x(\bullet_1 \psi)) = \{\bullet_1 \text{ch}_x(\psi, \text{tr}_x(\psi))\} \cup \text{SubF}(\text{ch}_x(\psi, \text{tr}_x(\psi)))$  and  $\text{SubBin}(\bullet_1 \text{tr}_x(\psi)) = \text{SubBin}(\text{tr}_x(\psi))$ . Thus,  $\mathcal{P}(\bullet_1 \psi)$  follows directly from  $\mathcal{P}(\psi)$ .

Assume now that  $\mathcal{P}(\psi_1), \mathcal{P}(\psi_2)$  hold for some ATL-formulas  $\psi_1, \psi_2$ . Consider any binary operator  $\bullet_2$ . We have  $\text{tr}_x(\psi \bullet_2 \psi') := \text{ch}_x(\psi, \text{tr}_x(\psi)) \bullet_2 \text{ch}_x(\psi', \text{tr}_x(\psi'))$ , hence:

$$\text{SubF}(\text{tr}_x(\psi \bullet_2 \psi')) = \{\text{tr}_x(\psi \bullet_2 \psi')\} \cup \text{SubF}(\text{ch}_x(\psi, \text{tr}_x(\psi))) \cup \text{SubF}(\text{ch}_x(\psi', \text{tr}_x(\psi')))$$

<sup>4</sup>Note that, to apply Lemma 64, it is crucial that  $\phi$  does not feature negations.

Since  $\text{SubF}(\text{ch}_x(\psi, \text{tr}_x(\psi))) \cap \text{SubF}(\phi_2) = \emptyset$  (by  $\mathcal{P}(\psi)$ ) and  $\text{SubF}(\text{ch}_x(\psi', \text{tr}_x(\psi'))) \cap \text{SubF}(\phi_2) = \emptyset$  (by  $\mathcal{P}(\psi')$ ), it follows that  $\text{SubF}(\text{tr}_x(\psi \bullet_2 \psi')) \cap \text{SubF}(\phi_2) = \emptyset$ . Furthermore, we have:

$$\text{SubBin}(\text{tr}_x(\psi \bullet_2 \psi')) = \{\text{tr}_x(\psi \bullet_2 \psi')\} \cup \text{SubBin}(\text{ch}_x(\psi, \text{tr}_x(\psi))) \cup \text{SubBin}(\text{ch}_x(\psi', \text{tr}_x(\psi')))$$

with  $\{\text{tr}_x(\psi \bullet_2 \psi')\} = \text{tr}_x(\{\psi \bullet_2 \psi'\})$ , and for any  $\tilde{\psi} \in \{\psi, \psi'\}$ , we have  $\text{SubBin}(\text{ch}_x(\tilde{\psi}, \text{tr}_x(\tilde{\psi}))) \subseteq \text{tr}_x(\text{SubBin}(\tilde{\psi}) \setminus \text{SubBin}(\phi_2))$  (by  $\mathcal{P}(\tilde{\psi})$ ). Thus,  $\mathcal{P}(\psi \bullet_2 \psi')$  follows from the fact that, if  $\psi \bullet_2 \psi' \notin \text{SubF}(\phi_2)$ , then  $\text{SubBin}(\psi \bullet_2 \psi') \setminus \text{SubBin}(\phi_2) = \{\psi \bullet_2 \psi'\} \cup (\text{SubBin}(\psi) \setminus \text{SubBin}(\phi_2)) \cup (\text{SubBin}(\psi') \setminus \text{SubBin}(\phi_2))$ . In fact,  $\mathcal{P}(\psi)$  holds for all ATL-formulas  $\psi$ .

Now, let  $\phi_1^x := \text{ch}_x(\phi_1, \text{tr}_x(\phi_1))$  and  $\phi^x := \phi_1^x \bullet \phi_2$ . By  $\mathcal{P}(\phi_1^x)$ , we have the following facts:

- $\text{SubF}(\phi_1^x) \cap \text{SubF}(\phi_2) = \emptyset$ ;
- $\text{SubBin}(\phi^x) = \{\phi^x\} \cup \text{SubBin}(\phi_1^x) \cup \text{SubBin}(\phi_2)$  and  $\text{SubBin}(\phi) = \{\phi\} \cup \text{SubBin}(\phi_1) \cup \text{SubBin}(\phi_2)$ . Since  $\text{SubBin}(\phi_1^x) \subseteq \text{tr}_x(\text{SubBin}(\phi_1) \setminus \text{SubBin}(\phi_2))$ , we have:

$$\begin{aligned} |\phi^x|_{\text{Bin}} &= |\text{SubBin}(\phi^x)| = 1 + |\text{SubBin}(\phi_1^x)| + |\text{SubBin}(\phi_2)| \\ &\leq 1 + |\text{SubBin}(\phi_1) \setminus \text{SubBin}(\phi_2)| + |\text{SubBin}(\phi_2)| \\ &= |\text{SubBin}(\phi)| = |\phi|_{\text{Bin}} \end{aligned}$$

That is,  $|\phi^x|_{\text{Bin}} \leq |\phi|_{\text{Bin}}$ :

- If  $\text{SubF}(\phi_1) \cap \text{SubF}(\phi_2) = \emptyset$ , then  $\phi^x = \phi$  and  $\text{Prop}(\phi) = \text{Prop}(\phi^x)$ . Otherwise, we have  $\text{Prop}(\phi) \cup \{x\} = \text{Prop}(\phi^x)$  (it can be proved straightforwardly by induction).

Therefore, we have  $|\phi^x|_{\text{Bin}} = |\phi_1^x|_{\text{Bin}} + |\phi_2|_{\text{Bin}} + 1 \leq k + 1$ . Hence, we can apply our induction hypotheses to  $|\phi_1^x|_{\text{Bin}} \leq k$  and  $|\phi_2|_{\text{Bin}} \leq k$  to obtain that:  $|\text{Prop}(\phi_1^x)| \leq |\phi_1^x|_{\text{Bin}} + 1$ ,  $|\text{Prop}(\phi_2)| \leq |\phi_2|_{\text{Bin}} + 1$ . Hence,  $|\text{Prop}(\phi)| \leq |\text{Prop}(\phi^x)| = |\text{Prop}(\phi_1^x)| + |\text{Prop}(\phi_2)| \leq k + 2$ . In addition, if  $\text{SubF}(\phi_1) \cap \text{SubF}(\phi_2) \neq \emptyset$ , then  $|\text{Prop}(\phi)| < |\text{Prop}(\phi^x)|$ , and thus  $|\text{Prop}(\phi)| < k + 2$ .

If instead there is an ATL-formula  $\phi' = \phi_1 \bullet \phi_2$ , for some ATL-formulas  $\phi_1, \phi_2$  and binary operator  $\bullet$ , such that  $\phi = \text{Qt} \cdot \phi'$  where  $\text{Qt}$  is a non-empty sequence of unary operators, we can apply the above arguments to the formula  $\phi'$ . Thus, the property holds also for  $k + 1$ . In fact, it holds for all  $k \in \mathbb{N}$ . The lemma follows.  $\square$

We can now proceed to the proof of Lemma 63.

*Proof.* First of all, note that for all ATL-formulas  $\phi$  such that  $\text{Prop}(\phi) \subseteq \text{Prop}'_n$ , and for all  $S \subseteq \text{Prop}'_n$ , there is  $\psi \in \{\text{True}, \text{False}\}$  such that we have  $\phi \equiv_{k,n,S} \psi$ . This can be straightforwardly established by induction on ATL-formulas  $\phi$ .

Now, let us show by induction on  $0 \leq i \leq n$  the property  $\mathcal{P}(i)$ : for all ATL-formulas  $\phi \in \text{ATL}^k(\text{Prop}, \text{U}^t, \emptyset, \text{B}^l, i)$  such that  $|\text{Prop}(\phi) \cap \text{Prop}'_n| = i$ , and for all  $S \subseteq \text{Prop}'_n$ , there are two ATL-formulas  $\psi, \hat{\psi} \in \text{ATL}^k(\{p, \bar{p}\}, \text{U}^t \setminus \{\neg\}, \emptyset, \text{B}^l, 0)$  such that:  $|\psi|, |\hat{\psi}| \leq |\phi| - 2i$ ,  $\phi \equiv_{k,n,S} \psi$ , and  $\neg\phi \equiv_{k,n,S} \hat{\psi}$ .

Consider first the case  $i = 0$ . Let  $\phi \in \text{ATL}^k(\text{Prop}, \text{U}^t, \emptyset, \text{B}^l, 0)$ . We have  $\phi := \text{Qt} \cdot r$  for some  $r \in \{p, \bar{p}\}$  and  $\text{Qt} \in (\text{Op}(k, \text{U}^t))^*$ . For  $j \in \{0, 1\}$ , we let  $(\text{Qt}_j, x_j) := \text{UnNeg}(\text{Qt}, j)$  and  $r_j \in \{p, \bar{p}\}$  be such that  $r_j = r$  if and only if  $x_j = 0$ . Then, we let  $\psi := \text{Qt}_0 \cdot r_0$  and  $\hat{\psi} := \text{Qt}_1 \cdot r_1$ . By Lemma 61, we have:  $\psi, \hat{\psi} \in \text{ATL}^k(\text{Prop}, \text{U}^t \setminus \{\neg\}, \emptyset, \text{B}^l, 0)$  and  $|\psi|, |\hat{\psi}| \leq |\phi|$ . Furthermore, we also have, for all  $S \subseteq \text{Prop}'_n$ ,  $\phi \equiv_{k,n,S} \psi$ , and  $\neg\phi \equiv_{k,n,S} \hat{\psi}$ , again by Lemma 61, and since  $p \equiv_{k,n,S} \neg\bar{p}$  (by definition of  $(n, S)$ -proper structures). Thus,  $\mathcal{P}(0)$  holds.

Assume now that it  $\mathcal{P}(j)$  holds for all  $j \leq i$ , for some  $i \leq n-1 \in \mathbb{N}$ . Consider an ATL-formula  $\phi \in \text{ATL}^k(\text{Prop}, \text{U}^t, \emptyset, \text{B}^l, i+1)$  such that  $|\text{Prop}(\phi) \cap \text{Prop}'_n| = i+1$ . Let  $S \subseteq \text{Prop}'_n$ . If  $|\text{Prop}(\phi)| =$

$i + 1$ , then we have  $|\phi| \geq 2i + 3$  by Lemma 12<sup>5</sup>. Furthermore, there is  $\psi, \widehat{\psi} \in \{\text{True}, \text{False}\}$  of size 1 such that,  $\phi \equiv_{k,n,S} \psi$  and  $\neg\phi \equiv_{k,n,S} \widehat{\psi}$ . Assume now that  $|\text{Prop}(\phi)| > i + 1$ . Since  $|\text{Prop}(\phi)| \leq |\phi|_{\text{Bin}} + 1 \leq i + 2$  (by Lemma 66), it follows that  $|\text{Prop}(\phi)| = i + 2$ . Let  $x \in \{p, \bar{p}\}$  be such that  $x \in \text{Prop}(\phi)$ . Then, there is some sequence of unary operators  $\text{Qt} \in (\text{Op}(k, \text{U}^t))^*$ , a binary operator  $\bullet$  and two ATL-formulas  $\phi_1, \phi_2 \in \text{ATL}^k(\text{Prop}, \text{U}^t, \emptyset, \text{B}^l, i)$  such that  $\phi = \text{Qt} \cdot (\phi_1 \bullet \phi_2)$ . Thus,  $\text{Prop}(\phi) = \text{Prop}(\phi_1 \bullet \phi_2)$ . Since we have  $\phi_1 \bullet \phi_2 \in \text{ATL}^k(\text{Prop}, \text{U}^t, \emptyset, \text{B}^l, i + 1)$  and  $|\text{Prop}(\phi_1 \bullet \phi_2)| = i + 2$ , it follows that  $\text{Prop}(\phi_1) \cap \text{Prop}(\phi_2) = \emptyset$ , by Lemma 66. Without loss of generality, let us assume that  $x \in \text{Prop}(\phi_1)$ . In that case, we have  $\text{Prop}(\phi_2) \subseteq \text{Prop}'_n$ . Therefore, as mentioned at the beginning of this proof, there is some  $\psi_{\text{tr}} \in \{\text{True}, \text{False}\}$  such that  $\phi_2 \equiv_{k,S} \psi_{\text{tr}}$ . Hence,  $\phi_1 \bullet \phi_2 \equiv_{k,S} \phi_1 \bullet \psi_{\text{tr}}$ . It follows, since  $\bullet$  is a binary logical operator, that there is some  $\Psi, \widehat{\Psi} \in \{\text{True}, \text{False}, \phi_1, \neg\phi_1\}$  such that  $\phi_1 \bullet \phi_2 \equiv_{k,n,S} \Psi$  and  $\neg(\phi_1 \bullet \phi_2) \equiv_{k,n,S} \widehat{\Psi}$ . Now, let  $m_1 := |\text{Prop}(\phi_1) \cap \text{Prop}'_n| \leq i$ . We have  $m_2 := |\text{Prop}(\phi_2)|$  such that  $m_1 + m_2 = |\text{Prop}(\phi)| = i + 1$ . In addition, by  $\mathcal{P}(m_1)$ ,  $\Psi$  and  $\widehat{\Psi}$  can be chosen such that  $\Psi, \widehat{\Psi} \in \text{ATL}^k(\{p, \bar{p}\}, \text{U}^t \setminus \{\neg\}, \emptyset, \text{B}^l, 0)$  and  $|\Psi|, |\widehat{\Psi}| \leq |\phi_1| - 2m_1$ . In addition, by Lemma 12, we have  $|\phi_2| \geq 2m_2 - 1$ . Overall, we obtain that  $\Psi, \widehat{\Psi}$  can be chosen such that (note that since  $\text{Prop}(\phi_1) \cap \text{Prop}(\phi_2) = \emptyset$ , we have  $\text{SubF}(\phi_1) \cap \text{SubF}(\phi_2) = \emptyset$ ):

$$\begin{aligned} |\Psi|, |\widehat{\Psi}| &\leq |\phi_1| - 2m_1 = |\phi_1 \bullet \phi_2| - (|\phi_2| + 1) - 2m_1 \\ &\leq |\phi_1 \bullet \phi_2| - 2m_2 - 2m_1 = |\phi_1 \bullet \phi_2| - 2(i + 1) \end{aligned}$$

Then, for  $j \in \{0, 1\}$ , we let  $(\text{Qt}_j, x_j) := \text{UnNeg}(\text{Qt}, j)$  and  $\psi_j \in \{\Psi, \widehat{\Psi}\}$  be such that  $\psi_j \equiv_{k,n,S} \Psi$  if and only if  $x_j = 0$ . Then, we let  $\psi := \text{Qt}_0 \cdot \psi_0$  and  $\widehat{\psi} := \text{Qt}_1 \cdot \psi_1$ . By Lemma 61, we have:  $\psi, \widehat{\psi} \in \text{ATL}^k(\text{Prop}, \text{U}^t \setminus \{\neg\}, \emptyset, \text{B}^l, 0)$ ,  $\phi \equiv_{k,n,S} \psi$  and  $\neg\phi \equiv_{k,n,S} \widehat{\psi}$ , and  $|\text{Qt}_0|, |\text{Qt}_1| \leq |\text{Qt}|$ . Thus, we have  $|\psi| = |\text{Qt}_0| + |\psi_0| \leq |\text{Qt}| + |\phi_1 \bullet \phi_2| - 2(i + 1) = |\phi| - 2(i + 1)$ , and similarly for  $\widehat{\psi}$ . Thus,  $\mathcal{P}(i + 1)$  follows. In fact,  $\mathcal{P}(i)$  holds for all  $0 \leq i \leq n$ . The lemma is then given by  $\mathcal{P}(n)$ .  $\square$

Let us consider Lemma 64. Before we proceed to its proof, we state below a useful lemma analogous to Lemma 11.

**Lemma 67.** *For all sets of propositions  $\text{Prop}$ , for all  $Y \subseteq \text{Prop}$ , and  $S, S' \subseteq \text{Prop}$ , if, for all  $x \in Y$ , we have  $x \in S$  if and only if  $x \in S'$ , then an ATL-formula  $\phi$  such that  $\text{Prop}(\phi) \subseteq Y$  cannot distinguish the trivial structures  $T(S)$  and  $T(S')$ .*

*Proof.* A straightforward proof by induction on ATL-formulas establishes the lemma.  $\square$

We can now proceed to the proof of Lemma 64.

*Proof.* Let  $n \in \mathbb{N}$ .

- Assume that  $\bullet = \vee$ . The cases  $\bullet = \implies$  and  $\bullet = \Leftarrow$  are analogous. We let  $S_{n,\bullet} := \emptyset$  and, for all  $1 \leq i \leq n$ , we let  $S_i := \{p_i\}$ . Then, we define:  $\mathcal{A}_{n,\bullet} := \{T(S_i) \mid 1 \leq i \leq n\}$ ,  $\mathcal{B}_{n,\bullet} := \{T(S_{n,\bullet})\}$ , and  $\phi_{n,\bullet} := p_1 \vee \dots \vee p_n$ .

That way, for all  $1 \leq i \leq n$ , to distinguish  $T(S_i)$  and  $T(S)$ , an ATL-formula  $\phi$  needs to be such that  $p_i \in \text{Prop}(\phi)$ , by Lemma 67. Hence, if an ATL-formula  $\phi$  separates  $\mathcal{A}_{n,\bullet}$  and  $\mathcal{B}_{n,\bullet}$ , we have  $\text{Prop}'_n \subseteq \text{Prop}(\phi)$ . Consider now any two positive and negative sets  $\mathcal{P}, \mathcal{N}$  of  $(n, S_{n,\bullet})$ -proper structures. Let  $\mathcal{P}' := \mathcal{P} \cup \mathcal{A}_{n,\bullet}$  and  $\mathcal{N}' := \mathcal{N} \cup \mathcal{B}_{n,\bullet}$ . Consider any formula  $\phi \in \text{ATL}(\text{Prop}^n, \text{U}^t, \emptyset, \text{B}^l, 0)$  and  $\psi := \phi \vee \phi_{n,\bullet}$ . For all  $1 \leq i \leq n$ , we have  $T(S_i) \models \phi_{n,\bullet}$  and  $T(S_{n,\bullet}) \not\models \phi_{n,\bullet}$ . Furthermore, since  $\phi$  does not use negations, we also have  $T(S_{n,\bullet}) \not\models \phi$ . In addition, by definition of  $S_{n,\bullet}$ , for all  $(n, S_{n,\bullet})$ -proper structures  $T$ , we have  $T \not\models \phi_{n,\bullet}$ . Thus, we have that  $\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$  if and only if  $\psi$  accepts  $\mathcal{P}'$  and rejects  $\mathcal{N}'$ .

<sup>5</sup>This lemma is established for LTL-formulas, but its result can be applied to ATL-formulas as well



- Assume now that  $\bullet = \wedge$ . The cases  $\bullet = \neg \Rightarrow$  and  $\bullet = \neg \Leftarrow$  are analogous. We let  $S_{n,\bullet} := \text{Prop}'_n$ ,  $S := \text{Prop}_n$  and, for all  $1 \leq i \leq n$ , we let  $S_i := \text{Prop}_n \setminus \{p_i\}$ . Then, we define:  $\mathcal{A}_{n,\bullet} := \{T(S)\}$ ,  $\mathcal{B}_{n,\bullet} := \{T(S_i) \mid 1 \leq i \leq n\}$ , and  $\phi_{n,\bullet} := p_1 \wedge \dots \wedge p_n$ .

That way, for all  $1 \leq i \leq n$ , to distinguish  $T(S_i)$  and  $T(S)$ , an ATL-formula  $\phi$  needs to be such that  $p_i \in \text{Prop}(\phi)$ , by Lemma 67. Hence, if an ATL-formula  $\phi$  separates  $\mathcal{A}_{n,\bullet}$  and  $\mathcal{B}_{n,\bullet}$ , we have  $\text{Prop}'_n \subseteq \text{Prop}(\phi)$ . Consider now any two positive and negative sets  $\mathcal{P}, \mathcal{N}$  of  $(n, S_{n,\bullet})$ -proper structures. Let  $\mathcal{P}' := \mathcal{P} \cup \mathcal{A}_{n,\bullet}$  and  $\mathcal{N}' := \mathcal{N} \cup \mathcal{B}_{n,\bullet}$ . Consider any formula  $\phi \in \text{ATL}(\text{Prop}^n, \text{U}^t, \emptyset, \mathbf{B}^1, 0)$  and  $\psi := \phi \wedge \phi_{n,\bullet}$ . We have  $T(S) \models \phi_{n,\bullet}$  and for all  $1 \leq i \leq n$ , we have  $T(S_i) \not\models \phi_{n,\bullet}$ . Furthermore, since  $\phi$  does not use negations, we also have  $T(S) \models \phi$ . In addition, by definition of  $S_{n,\bullet}$ , for all  $(n, S_{n,\bullet})$ -proper structures  $T$ , we have  $T \models \phi_{n,\bullet}$ . Overall, we obtain that  $\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$  if and only if  $\psi$  accepts  $\mathcal{P}'$  and rejects  $\mathcal{N}'$ .

- Assume now that  $\bullet = \neg \wedge$ , the case  $\bullet = \neg \vee$  is analogous. We let  $\phi^1 := p_1$  and, for all  $2 \leq i \leq n$ , we let  $\phi^i := \phi^{i-1} \neg \wedge p_i$ . Let us first prove by induction on  $1 \leq i \leq n$  that, for all  $S \subseteq \{p_1, \dots, p_i\}$ , letting  $\tilde{S}^i := S \cup \{p_{i+1}, \dots, p_n\}$ , we have that  $T(S) \models \phi^i$  and  $T(\tilde{S}^i) \models \phi^n$  have the same truth value if and only if  $i$  and  $n$  have the same parity. This obviously holds for  $i = n$ . Assume now that it holds for some  $2 \leq i \leq n$ . Let  $S \subseteq \{p_1, \dots, p_{i-1}\}$ . We have  $\phi^i := \phi^{i-1} \neg \wedge p_i \equiv \neg \phi^{i-1} \vee \neg p_i$ . Since  $p_i \in \tilde{S}^{i-1}$ , and  $\text{Prop}(\phi^{i-1}) \subseteq \{p_1, \dots, p_{i-1}\}$  we have  $T(\tilde{S}^{i-1}) \models \phi^i$  if and only if  $T(S \cup \{p_i\}) \models \phi^i$  if and only if  $T(S) \not\models \phi^{i-1}$ . By our induction hypothesis, we have that  $T(S \cup \{p_i\}) \models \phi^i$  and  $T(\widetilde{S \cup \{p_i\}}^i) \models \phi^n$  have the same truth value if and only if  $i$  and  $n$  have the same parity. We deduce that  $T(S) \models \phi^{i-1}$  and  $T(\tilde{S}^{i-1}) \models \phi^n$  have the same truth value if and only if  $i-1$  and  $n$  have the same parity, since  $\widetilde{S \cup \{p_i\}}^i = \tilde{S}^{i-1}$ . In fact, the property holds for all  $1 \leq i \leq n$ .

Let us now show by induction on  $1 \leq i \leq n$  that there is some  $S_i, S'_i \subseteq \{p_1, \dots, p_i\}$  such that  $S_i \cup \{p_i\} = S'_i \cup \{p_i\}$ , and  $T(\tilde{S}_i^i) \models \phi^n$  while  $T(\tilde{S}'_i^i) \not\models \phi^n$ . We have  $T(\{p_1\}) \models \phi^1$  and  $T(\emptyset) \not\models \phi^1$ . Hence, we can conclude that this property holds for  $i = 1$  with what we have proved above. Assume now that it holds for some  $1 \leq i-1 \leq n-1$ . We have  $\phi^i := \phi^{i-1} \neg \wedge p_i$ . We let  $S := S_{i-1}$  if  $n$  and  $i-1$  have the same parity, and  $S := S'_{i-1}$  otherwise. That way, with our above result and the induction hypothesis, we know that  $T(S) \models \phi^{i-1}$ . Then, for all  $S_i, S'_i \in \{S \cup \{p_i\}, S\}$  such that  $S_i \neq S'_i$ , we have that the truth value of  $T(S_i) \models \phi^i$  and  $T(S'_i) \models \phi^i$  are different. We can then conclude with our above result. In fact, this property holds for all  $1 \leq i \leq n$ .

We can now finally define the formula and structures that we consider. We let  $\phi_{n,\bullet} := \phi^n$  and  $S_{n,\bullet} := S_n \subseteq \{p_1, \dots, p_n\}$ . Furthermore, we let  $\mathcal{A}_{n,\bullet} := \{T(\tilde{S}_i^i \cup \{p, \bar{p}\}) \mid 1 \leq i \leq n\}$ , and  $\mathcal{B}_{n,\bullet} := \{T(\tilde{S}'_i^i \cup \{p, \bar{p}\}) \mid 1 \leq i \leq n\}$ .

That way, for all  $1 \leq i \leq n$ , to distinguish  $T(\tilde{S}_i^i \cup \{p, \bar{p}\})$  and  $T(\tilde{S}'_i^i \cup \{p, \bar{p}\})$ , an ATL-formula  $\phi$  needs to be such that  $p_i \in \text{Prop}(\phi)$ , since  $S_i \cup \{p_i\} = S'_i \cup \{p_i\}$  and by Lemma 67. Hence, if an ATL-formula  $\phi$  separates  $\mathcal{A}_{n,\bullet}$  and  $\mathcal{B}_{n,\bullet}$ , we have  $\text{Prop}'_n \subseteq \text{Prop}(\phi)$ . Consider now any two positive and negative sets  $\mathcal{P}, \mathcal{N}$  of  $(n, S_{n,\bullet})$ -proper structures. Let  $\mathcal{P}' := \mathcal{N} \cup \mathcal{B}_{n,\bullet}$  and  $\mathcal{N}' := \mathcal{P} \cup \mathcal{A}_{n,\bullet}$ . Consider any formula  $\phi \in \text{ATL}(\text{Prop}^n, \text{U}^t, \emptyset, \mathbf{B}^1, 0)$  and  $\psi := \phi \neg \wedge \phi_{n,\bullet} \equiv \neg \phi \vee \neg \phi_{n,\bullet}$ . Let  $1 \leq l \leq n$ . We have  $\phi_{n,\bullet} \models T(\tilde{S}_i^i)$  and  $\phi \models T(\tilde{S}_i^i)$ . Therefore,  $\psi \not\models T(\tilde{S}_i^i)$ . However,  $\phi_{n,\bullet} \not\models T(\tilde{S}'_i^i)$ . Therefore,  $\psi \models T(\tilde{S}'_i^i)$ . Hence,  $\psi$  accepts  $\mathcal{B}_{n,\bullet}$  and rejects  $\mathcal{A}_{n,\bullet}$ . In addition, by definition of  $S_{n,\bullet}$ , for all  $(n, S_{n,\bullet})$ -proper structures  $T$ , we have  $T \models \phi_{n,\bullet}$ . Overall, we obtain that  $\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$  if and only if  $\psi$  accepts  $\mathcal{P}'$  and rejects  $\mathcal{N}'$ .

- Assume now that  $\bullet = \Leftrightarrow$ . The case  $\bullet = \neg \Leftrightarrow$  is analogous. We let  $S_{n,\bullet} := \text{Prop}'_n$ ,  $S := \text{Prop}_n$  and, for all  $1 \leq i \leq n$ , we let  $S_i := \text{Prop}_n \setminus \{p_i\}$ . Then, we define:  $\mathcal{A}_{n,\bullet} := \{T(S)\}$ ,  $\mathcal{B}_{n,\bullet} := \{T(S_i) \mid 1 \leq i \leq n\}$ , and  $\phi_{n,\bullet} := p_1 \Leftrightarrow \dots \Leftrightarrow p_n$ .

That way, for all  $1 \leq i \leq n$ , to distinguish  $T(S_i)$  and  $T(S)$ , an ATL-formula  $\phi$  needs to be such that  $p_i \in \text{Prop}(\phi)$ , by Lemma 67. Hence, if an ATL-formula  $\phi$  separates  $\mathcal{A}_{n,\bullet}$  and  $\mathcal{B}_{n,\bullet}$ , we have  $\text{Prop}'_n \subseteq \text{Prop}(\phi)$ . Consider now any two positive and negative sets  $\mathcal{P}, \mathcal{N}$  of  $(n, S_{n,\bullet})$ -proper structures. Let  $\mathcal{P}' := \mathcal{P} \cup \mathcal{A}_{n,\bullet}$  and  $\mathcal{N}' := \mathcal{N} \cup \mathcal{B}_{n,\bullet}$ . Consider any formula  $\phi \in \text{ATL}(\text{Prop}^n, \text{U}^t, \emptyset, \mathbf{B}^l, 0)$  and  $\psi := \phi \Leftrightarrow \phi_{n,\bullet}$ . We have  $T(S) \models \phi_{n,\bullet}$  and for all  $1 \leq i \leq n$ , we have  $T(S_i) \not\models \phi_{n,\bullet}$ . Furthermore, since  $\phi$  does not use negations, we also have  $T(S) \models \phi$ , and  $T(S_i) \models \phi$  for all  $1 \leq i \leq n$ . Thus,  $\psi$  accepts  $\mathcal{A}_{n,\bullet}$  and rejects  $\mathcal{B}_{n,\bullet}$ . In addition, by definition of  $S_{n,\bullet}$ , for all  $(n, S_{n,\bullet})$ -proper structures  $T$ , we have  $T \models \phi_{n,\bullet}$ . Thus, we have that  $\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$  if and only if  $\psi$  accepts  $\mathcal{P}'$  and rejects  $\mathcal{N}'$ . □

### 4.3 CTL learning

We start with CTL learning. We consider two cases: with and without the next operator  $\mathbf{X}$ . In the former case, the learning problem is NP-complete, in the latter it is NL-complete.

#### 4.3.1 With the next operator $\mathbf{X}$

The goal of this subsection is to show the theorem below.

**Theorem 68.** *Consider a set  $\text{U}^t \subseteq \text{Op}_{\text{Un}}$  of unary temporal operators and assume that  $\mathbf{X} \in \text{U}^t$ . Then, for all sets  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ , the decision problem  $\text{CTL}_{\text{Learn}}(\text{U}^t, \emptyset, \mathbf{B}^l, n)$  is NP-complete.*

For the remainder of this subsection, we consider a set  $\text{U}^t \subseteq \text{Op}_{\text{Un}}$  of unary temporal operators and assume that  $\mathbf{X} \in \text{U}^t$ .

**Overview of the reduction.** We follow the steps described in Section 4.2.1. However, Step a) was already taken care of in the previous section. This lets us focus on CTL-formulas using only unary operators (and a single proposition). First of all, we define Kripke structures that prevent the use of the  $\mathbf{F}$  and  $\mathbf{G}$  operators, as well as negations. Hence, the only operators remaining are  $\forall \mathbf{X}$  and  $\exists \mathbf{X}$ . Our idea is that now a CTL-formula of size  $l+1$  is entirely defined by a subset  $H \subseteq [1, \dots, l]$ : such a subset defines the CTL-formula  $\phi(l, H)$  in which, for all  $i \in [1, \dots, l]$ , the  $i$ -th operator of  $\phi$  is  $\exists \mathbf{X}$  if and only if  $i \in H$ .

On the other hand, a subset  $C \subseteq [1, \dots, l]$  defines a positive Kripke structure  $K^{l,C}$  that is defined as a sequence of  $l+1$  states  $\{q_1^{\text{lose}}, \dots, q_{l+1}^{\text{lose}}\}$  such that no CTL-formula of the shape  $\phi(l, H)$  can satisfy the state  $q_{l+1}^{\text{lose}}$ . Furthermore, all states  $q_i^{\text{lose}}$  are such that  $q_{i+1}^{\text{lose}} \in \text{Succ}(q_i^{\text{lose}})$ . However, some states  $q_i^{\text{lose}}$  can branch out to the winning state  $q^{\text{win}}$  (that all CTL-formulas of the shape  $\phi(l, H)$  satisfy). This occurs for those indices  $i \in [1, \dots, l]$  such that  $i \in C$ . In fact, with such a definition, we obtain that  $K^{l,C} \models \phi(l, H)$  if and only if  $H \cap C \neq \emptyset$ .

The final step that we take is to define, for  $k \leq l$ , a negative Kripke structure  $K_{\exists > k}^l$  that is satisfied by the CTL-formula  $\phi(l, H)$  if and only if  $\phi(l, H)$  uses at least  $k+1$  times the operator  $\exists \mathbf{X}$ , which is equivalent to  $|H| > k$ . The Kripke structure  $K_{\exists > k}^l$  has  $k+2$  levels, from the bottom up:

- with a single starting state at the bottommost level;

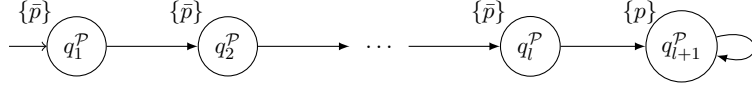


Figure 2: The Kripke structure  $K_l^P$ .

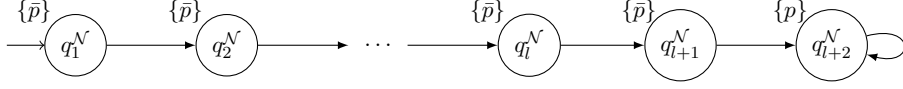


Figure 3: The Kripke structure  $K_l^N$ .

- where the formula  $p$  is satisfied in no state of the bottom  $k + 1$  levels, but it is satisfied in the only (self-looping) state  $q^{\text{win}}$  of the topmost level;
- where every state of the bottom  $k + 1$  levels has a successor one level higher.

**Formal definitions and proofs.** Let us now formally define the Kripke structures that we will use in the reduction. We first define the reduction for  $n = 0$  (i.e. when no binary operator is allowed) and prove its correctness. We then use it to exhibit a reduction for arbitrary  $n \in \mathbb{N}$ .

We start with the Kripke structures that ensure that the candidate formulas are of a specific shape.

**Definition 69.** Given  $l \in \mathbb{N}_1$  we define the two Kripke structures  $K_l^P$  and  $K_l^N$  that can be seen in Figures 2 and 3.

Let us now consider the Kripke structures  $K^{l,C}$  that encode a subset of  $C \subseteq [1, \dots, l]$ .

**Definition 70.** Given some  $l \in \mathbb{N}_1$  and  $C \subseteq [1, \dots, l]$ , we define the Kripke structure  $K_{(l,C)} = \langle Q, I, \{p, \bar{p}\}, \pi, \text{Succ} \rangle$  where:

- $Q := \{q_1^{\text{lose}}, \dots, q_{l+1}^{\text{lose}}, q^{\text{win}}\}$ ;
- $I := \{q_1^{\text{lose}}\}$ ;
- for all  $1 \leq i \leq l$ , we have:

$$\text{Succ}(q_i^{\text{lose}}) := \begin{cases} \{q_{i+1}^{\text{lose}}, q^{\text{win}}\} & \text{if } i \in C \\ \{q_{i+1}^{\text{lose}}\} & \text{if } i \notin C \end{cases}$$

Furthermore,  $\text{Succ}(q^{\text{win}}) := \{q^{\text{win}}\}$  and  $\text{Succ}(q_{l+1}^{\text{lose}}) := \{q_{l+1}^{\text{lose}}\}$ .

- $\pi(q^{\text{win}}) := \{p\}$  and, for all  $q \in Q \setminus \{q^{\text{win}}\}$ , we have  $\pi(q) := \{\bar{p}\}$ .

An example of such a construction is depicted in Figure 4.

Finally, we also consider a Kripke structure that prevents from using too many  $\exists$  operators.

**Definition 71.** Given some  $l \in \mathbb{N}_1$  and  $k \leq l$ , we define the Kripke structure  $K_{\exists > k}^l = \langle Q, I, \{p, \bar{p}\}, \pi, \text{Succ} \rangle$  where:

- $Q := \{q^{\text{win}}\} \cup \{q_{i,j}^{\text{lose}} \mid 0 \leq i \leq k, i + 1 \leq j \leq l + 1\}$ ;
- $I := \{q_{0,1}^{\text{lose}}\}$ ;

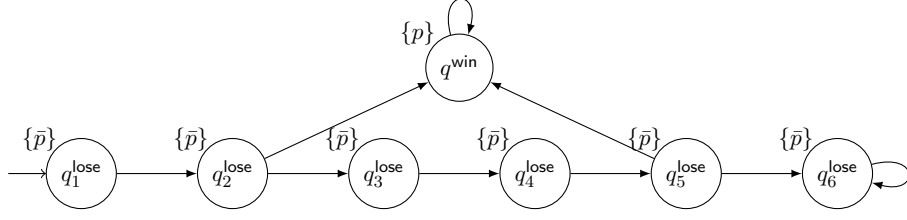


Figure 4: The Kripke structure  $K_{(5, \{2,5\})}$ .

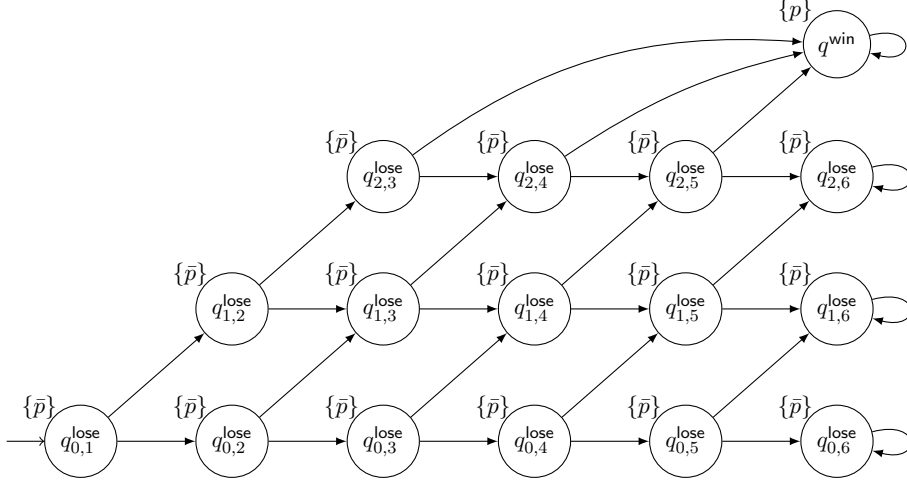


Figure 5: The Kripke structure  $K_{\exists_{>2}^5}$ .

- For all  $0 \leq i \leq k$ ,  $i+1 \leq j \leq l$ , we have:

$$\text{Succ}(q_{i,j}^{\text{lose}}) := \begin{cases} \{q_{i,j+1}^{\text{lose}}, q_{i+1,j+1}^{\text{lose}}\} & \text{if } i \leq k-1 \\ \{q_{i,j+1}^{\text{lose}}, q^{\text{win}}\} & \text{if } i = k \end{cases}$$

For all  $0 \leq i \leq k$ :

$$\text{Succ}(q_{i,l+1}^{\text{lose}}) := \{q_{i,l+1}^{\text{lose}}\}$$

and  $\text{Succ}(q^{\text{win}}) := \{q^{\text{win}}\}$ .

- $\pi(q^{\text{win}}) := \{p\}$  and, for all  $q \in Q \setminus \{q^{\text{win}}\}$ , we have  $\pi(q) := \bar{p}$ .

An example of such a Kripke structure is depicted in Figure 5.

We can finally define the reduction from the hitting set problem that we consider.

**Definition 72.** Consider an instance  $(l, C, k)$  of the hitting set problem Hit. We consider:

- $\text{Prop}_0 = \{p, \bar{p}\}$  as set of propositions;
- $\mathcal{P} := \{K_{(l, C_i)} \mid 1 \leq i \leq n\} \cup \{K_l^{\mathcal{P}}\}$ ;
- $\mathcal{N} := \{K_{\exists_{>k}^l}, K_l^{\mathcal{N}}\}$ ;
- $B := l+1$ .

Then, we define the input  $\text{In}_{(l,C,k)}^{\text{CTL}(\mathbf{X}),0} := (\text{Prop}, \mathcal{P}, \mathcal{N}, B)$ .

This reduction satisfies the lemma below.

**Lemma 73.** *The input  $(l, C, k)$  is a positive instance of the hitting set problem  $\text{Hit}$  if and only if  $\text{In}_{(l,C,k)}^{\text{CTL}(\mathbf{X}),0}$  is a positive instance of  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t \setminus \{-\}, \emptyset, \emptyset, 0)$  if and only if  $\text{In}_{(l,C,k)}^{\text{CTL}(\mathbf{X}),0}$  is a positive instance of  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t \cup \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ .*

We start with the formal definition below  $\phi(l, H)$ -formulas.

**Definition 74.** *Let  $l \in \mathbb{N}$ . A CTL-formula  $\phi \in \text{CTL}(\{p, \bar{p}\}, \mathbf{U}^t \cup \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$  is an  $\mathbf{X}^l$ -formula if  $\phi \in \text{CTL}(\{p, \bar{p}\}, \{\mathbf{X}\}, \emptyset, \emptyset, 0)$  and  $|\phi| = l + 1$ .*

*The  $\mathbf{X}^l$ -formulas differ by the indices where  $\exists$  and  $\forall$  quantifiers appear. For all  $H \subseteq [1, \dots, l]$ , we let  $\phi(l, H)$  denote the  $\mathbf{X}^l$ -formula:*

$$\phi(l, H) := Q_1 \mathbf{X} \dots Q_l \mathbf{X} p$$

where, for all  $i \in [1, \dots, l]$ , we have:

$$Q_i := \begin{cases} \exists & \text{if } i \in H \\ \forall & \text{if } i \in [1, \dots, l] \setminus H \end{cases}$$

Let us show that we can restrict ourselves to  $\mathbf{X}^l$ -formulas by considering the Kripke structures  $K_i^{\mathcal{P}}, K_i^{\mathcal{N}}$ .

**Lemma 75.** *Consider some  $l \in \mathbb{N}_1$  and a CTL formula  $\phi \in \text{CTL}(\{p, \bar{p}\}, \mathbf{U}^t, \emptyset, \emptyset, 0)$  of size at most  $l + 1$ . The formula  $\phi$  accepts  $K_i^{\mathcal{P}}$  and rejects  $K_i^{\mathcal{N}}$  if and only if  $\phi$  is an  $\mathbf{X}^l$ -formula.*

*Proof.* First of all, note that, for all  $1 \leq i \leq l + 1$ , the states  $q_i^{\mathcal{P}}$  and  $q_{i+1}^{\mathcal{N}}$  satisfy exactly the same CTL-formulas. Then, we show by induction on  $0 \leq i \leq l$  the property  $\mathcal{P}(i)$ :

- for all  $1 \leq j \leq l - i$ , any  $\text{CTL}(\{p, \bar{p}\}, \mathbf{U}^t, \emptyset, \emptyset, 0)$ -formula of size at most  $i + 1$  cannot distinguish the states  $q_j^{\mathcal{P}}$  and  $q_j^{\mathcal{N}}$ ;
- a  $\text{CTL}(\{p, \bar{p}\}, \mathbf{U}^t, \emptyset, \emptyset, 0)$ -formula of size at most  $i + 1$  distinguishes the states  $q_{l+1-i}^{\mathcal{P}}$  and  $q_{l+1-i}^{\mathcal{N}}$  if and only if it is an  $\mathbf{X}^i$ -formula (in which case, it accepts  $q_{l+1-i}^{\mathcal{P}}$  and rejects  $q_{l+1-i}^{\mathcal{N}}$ ).

The property  $\mathcal{P}(0)$  straightforwardly holds.

Assume now that the property  $\mathcal{P}(i)$  holds for some  $0 \leq i \leq l - 1$ . Consider a  $\text{CTL}(\{p, \bar{p}\}, \mathbf{U}^t, \emptyset, \emptyset, 0)$ -formula  $\phi$  of size at most  $i + 2$ . First of all, if  $\phi$  is of size at most  $i + 1$ , then  $\mathcal{P}(i)$  gives that for all  $1 \leq j \leq l + 1 - (i + 1) = l - i$ , the formula  $\phi$  does not distinguish the states  $q_j^{\mathcal{P}}$  and  $q_j^{\mathcal{N}}$ . Assume now that  $|\phi| = i + 2$ . Consider any  $1 \leq j \leq l - i$ . There are several cases.

- Assume that  $\phi = \neg\phi'$ , in which case  $|\phi'| = i + 1$ . By  $\mathcal{P}(i)$ ,  $\phi'$  does not distinguish the states  $q_j^{\mathcal{P}}$  and  $q_j^{\mathcal{N}}$ , thus  $\phi$  does not either.
- Assume that  $\phi = \mathbf{Q} \mathbf{F} \phi'$ , with  $\mathbf{Q} \in \{\exists, \forall\}$ , in which case  $|\phi'| = i + 1$ . Assume that  $q_j^{\mathcal{P}} \models \phi$ . Then, there is some  $j \leq r \leq l + 1$  such that  $q_r^{\mathcal{P}} \models \phi'$ , which is equivalent to  $q_{r+1}^{\mathcal{N}} \models \phi'$ , and thus  $q_j^{\mathcal{N}} \models \phi$ . On the other hand, assume that  $q_j^{\mathcal{N}} \models \phi$ , in which case there is some  $j \leq r \leq l + 2$  such that  $q_r^{\mathcal{N}} \models \phi'$ . If  $r \geq j + 1$ , then we have  $r - 1 \geq j$  and  $q_{r-1}^{\mathcal{P}} \models \phi'$ , and thus  $q_j^{\mathcal{P}} \models \phi$ . Otherwise,  $r = j$ . Since, by  $\mathcal{P}(i)$ , we have that  $\phi'$  does not distinguish  $q_j^{\mathcal{P}}$  and  $q_j^{\mathcal{N}}$ , then it follows that we also have  $q_j^{\mathcal{P}} \models \phi'$ , and thus  $q_j^{\mathcal{P}} \models \phi$ . Hence, in any case the formula  $\phi$  does not distinguish  $q_j^{\mathcal{P}}$  and  $q_j^{\mathcal{N}}$ .

- Assume that  $\phi = \mathbf{Q} \mathbf{G} \phi'$ , with  $\mathbf{Q} \in \{\exists, \forall\}$ . This case is analogous to the previous one.
- Assume finally that  $\phi = \mathbf{Q} \mathbf{X} \phi'$ , with  $\mathbf{Q} \in \{\exists, \forall\}$ , in which case  $|\phi'| = i + 1$ . If  $j < l - i$ , then  $j + 1 \leq l - i$ , and thus, by  $\mathcal{P}(i)$ ,  $\phi'$  does not distinguish the two states  $q_{j+1}^{\mathcal{P}}$  and  $q_{j+1}^{\mathcal{N}}$ . Hence,  $\phi$  does not distinguish the two states  $q_j^{\mathcal{P}}$  and  $q_j^{\mathcal{N}}$ . If  $j = l - i$ , then  $j + 1 = l + 1 - i$ , hence  $\phi'$  distinguishes the states  $q_{j+1}^{\mathcal{P}}$  and  $q_{j+1}^{\mathcal{N}}$  if and only if  $\phi'$  is an  $\mathbf{X}^i$ -formula (in which case it accepts  $q_{j+1}^{\mathcal{P}}$  and rejects  $q_{j+1}^{\mathcal{N}}$ ), which is equivalent to  $\phi$  being an  $\mathbf{X}^{i+1}$ -formula (in which case it accepts  $q_j^{\mathcal{P}}$  and rejects  $q_j^{\mathcal{N}}$ ).

Therefore, the property  $\mathcal{P}(i + 1)$  holds. In fact, it holds for all  $0 \leq i \leq l$ . The lemma is then given by  $\mathcal{P}(l)$ .  $\square$

We deduce as a corollary.

**Corollary 76.** *The input  $\text{In}_{(l,C,k)}^{\text{CTL}(\mathbf{X}),0}$  is a positive instance of  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$  if and only if  $\text{In}_{(l,C,k)}^{\text{CTL}(\mathbf{X}),0}$  is a positive instance of  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t \cup \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ .*

*Proof.* This is straightforward consequence of Lemma 73: if a formula  $\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ , then it accepts  $K_l^{\mathcal{P}}$  and rejects  $K_l^{\mathcal{N}}$ . Thus, if its size is at most  $l + 1$ , it only uses the operator  $\mathbf{X} \in \mathbf{U}^t$ .  $\square$

Let us now consider at which conditions the CTL  $\mathbf{X}^l$ -formulas  $\phi(l, H)$  accepts the Kripke structure  $K_{(l,C)}$ .

**Lemma 77.** *Consider some  $l \in \mathbb{N}_1$  and  $H, C \subseteq [1, \dots, l]$ . The CTL  $\mathbf{X}^l$ -formula  $\phi(l, H)$  accepts the Kripke structure  $K_{(l,C)}$  if and only if  $H \cap C \neq \emptyset$ .*

*Proof.* Let  $H \subseteq [1, \dots, l]$ . We have  $\phi(l, H) = Q_1 \mathbf{X} \dots Q_l \mathbf{X} p$ . For all  $1 \leq i \leq l + 1$ , we let  $H_i := H \cap [i, \dots, l]$ ,  $\phi_i(H) := Q_i \mathbf{X} \dots Q_l \mathbf{X} p$ . In particular,  $\phi_{l+1}(H) = p$  and  $\phi_1(H) = \phi(l, H)$ .

Let us show by induction on  $l + 1 \geq i \geq 1$  the property  $\mathcal{P}(i)$ :  $q_i^{\text{lose}} \models \phi_i(H)$  if and only if  $H_i \cap C \neq \emptyset$ . The property  $\mathcal{P}(l + 1)$  straightforwardly holds since  $\pi(q_{l+1}^{\text{lose}}) = \emptyset$  and  $H_{l+1} = \emptyset$ .

Assume now that  $\mathcal{P}(i)$  holds for some  $l + 1 \geq i \geq 2$ . We have  $\phi_{i-1}(H) := Q_{i-1} \mathbf{X} \phi_i(H)$  with  $Q_{i-1} = \exists$  if and only if  $i - 1 \in H$ . Then:

- Assume that  $H_{i-1} \cap C \neq \emptyset$ . If we have  $H_i \cap C \neq \emptyset$ , then by  $\mathcal{P}(i)$ ,  $q_i^{\text{lose}} \models \phi_i(H)$ . Since  $q_i^{\text{win}} \models \phi_i(H)$  and  $\text{Succ}(q_{i-1}^{\text{lose}}) \subseteq \{q_i^{\text{lose}}, q_i^{\text{win}}\}$ , it follows that  $q_{i-1}^{\text{lose}} \models \phi_{i-1}(H)$  (regardless of whether  $Q_{i-1} = \exists$  or  $Q_{i-1} = \forall$ ). Otherwise, it must be that  $i - 1 \in H \cap C$ . Thus, we have  $Q_{i-1} = \exists$  and  $q_i^{\text{win}} \in \text{Succ}(q_{i-1}^{\text{lose}})$ . Since  $q_i^{\text{win}} \models \phi_i(H)$ , it follows that  $q_{i-1}^{\text{lose}} \models \phi_{i-1}(H)$ .
- Assume now that  $H_{i-1} \cap C = \emptyset$ . It follows that  $q_i^{\text{lose}} \not\models \phi_i(H)$  by  $\mathcal{P}(i)$ , since  $H_i \cap C = \emptyset$ . There are two cases:
  - Assume that  $i - 1 \notin H$ . In that case,  $Q_{i-1} = \forall$ . Furthermore,  $q_i^{\text{lose}} \in \text{Succ}(q_{i-1}^{\text{lose}})$  and  $q_i^{\text{lose}} \not\models \phi_i(H)$ . Hence,  $q_{i-1}^{\text{lose}} \not\models \phi_{i-1}(H)$ .
  - Assume that  $i - 1 \notin C$ . In that case,  $\text{Succ}(q_{i-1}^{\text{lose}}) = \{q_i^{\text{lose}}\}$  and  $q_i^{\text{lose}} \not\models \phi_i(H)$ . Hence,  $q_{i-1}^{\text{lose}} \not\models \phi_{i-1}(H)$ .

Therefore, the property  $\mathcal{P}(i - 1)$  holds. In fact,  $\mathcal{P}(i)$  holds for all  $1 \leq i \leq l + 1$ . The lemma is then given by  $\mathcal{P}(1)$ .  $\square$

In addition, for a CTL  $\mathbf{X}^l$ -formula  $\phi(l, H)$  not to accept the Kripke structure  $K_{\exists > k}^l$ , it must have not too much existential quantifiers, as stated below.

**Lemma 78.** Consider some  $l \in \mathbb{N}_1$ ,  $k \leq l$  and  $H \subseteq [1, \dots, l]$ . The CTL  $\mathbf{X}^l$ -formula  $\phi(l, H)$  accepts the Kripke structure  $K_{\exists > k}^l$  if and only if  $|H| > k$ .

*Proof.* Let  $H \subseteq [1, \dots, l]$ . As in the proof of Lemma 77, we have  $\phi(l, H) = Q_1 \mathbf{X} \dots Q_l \mathbf{X} p$  and, for all  $1 \leq i \leq l+1$ , we let  $H_i := H \cap [i, \dots, l]$  and  $\phi_i(H) := Q_i \mathbf{X} \dots Q_l \mathbf{X} p$ . Thus,  $\phi_{l+1}(H) = p$  and  $\phi_1(H) = \phi(H)$ . Furthermore, for all  $k+1 \geq i \geq 0$ ,  $l+1 \geq j \geq i+1$ , we let:

$$q(i, j) := \begin{cases} q_{(i,j)}^{\text{lose}} & \text{if } i < k+1 \\ q^{\text{win}} & \text{otherwise} \end{cases}$$

Note that, for all  $k \geq i \geq 0$ ,  $l \geq j \geq i+1$ , we have  $\text{Succ}(q(i, j)) = \{q(i, j+1), q(i+1, j+1)\}$ .

Let us show by induction on  $k+1 \geq i \geq 0$  the property  $\mathcal{P}(i)$ : for all  $l+1 \geq j \geq i+1$ ,  $q(i, j) \models \phi_j(H)$  if and only if  $|H_j| > k-i$ .

The property  $\mathcal{P}(k+1)$  states that for all  $k+2 \leq j \leq l+1$ ,  $q(k+1, j) = q^{\text{win}} \models \phi_j(H)$  if and only if  $|\emptyset| > -1$ . This straightforwardly holds since  $\phi_j(H)$  is a  $\neg$ -free formula.

Assume now that the property  $\mathcal{P}(i)$  holds for some  $k+1 \geq i \geq 1$ . Let us show by induction on  $l+1 \geq j \geq i$  the property  $\mathcal{Q}(j)$ :  $q(i-1, j) \models \phi_j(H)$  if and only if  $|H_j| > k-i+1$ . The property  $\mathcal{Q}(l+1)$  straightforwardly holds since  $q(i-1, l+1) = q_{i-1, l+1}^{\text{lose}} \not\models \phi_{l+1}(H)$  and  $H_{l+1} = \emptyset$ . Assume now that the property  $\mathcal{Q}(j)$  holds for some  $l+1 \geq j \geq i+1$ . We have  $\phi_{j-1}(H) := Q_{j-1} \mathbf{X} \phi_j(H)$  with  $Q_{j-1} = \exists$  if and only if  $j-1 \in H$ .

- Assume that  $|H_{j-1}| > k-i+1$ . If we have  $j-1 \in H$ , then  $Q_{j-1} = \exists$  and  $|H_j| > k-i$ . Therefore, by  $\mathcal{P}(i)$ , we have  $q(i, j) \models \phi_j(H)$ . Since  $q(i, j) \in \text{Succ}(q(i-1, j-1))$ , it follows that  $q(i-1, j-1) \models \phi_{j-1}(H)$ .

On the other hand, if  $j-1 \notin H$ , then  $|H_j| > k-i+1 > k-i$ . Hence, by  $\mathcal{P}(i)$ , we have  $q(i, j) \models \phi_j(H)$  and by  $\mathcal{Q}(j)$ , we have  $q(i-1, j) \models \phi_j(H)$ . Since  $\text{Succ}(q(i-1, j-1)) = \{q(i, j), q(i-1, j)\}$ , it follows that  $q(i-1, j-1) \models \phi_{j-1}(H)$ .

- Assume now that  $|H_{j-1}| \leq k-i+1$ . If we have  $j-1 \in H$ , then  $|H_j| \leq k-i < k-i+1$ . Hence, by  $\mathcal{P}(i)$ , we have  $q(i, j) \not\models \phi_j(H)$  and by  $\mathcal{Q}(j)$ , we have  $q(i-1, j) \not\models \phi_j(H)$ . Since  $\text{Succ}(q(i-1, j-1)) = \{q(i, j), q(i-1, j)\}$ , it follows that  $q(i-1, j-1) \not\models \phi_{j-1}(H)$ .

On the other hand, if  $j-1 \notin H$ , then  $Q_{j-1} = \forall$  and  $|H_j| \leq k-i+1$ . Therefore, by  $\mathcal{Q}(j)$ , we have  $q(i-1, j) \not\models \phi_j(H)$ . Since  $q(i-1, j) \in \text{Succ}(q(i-1, j-1))$ , it follows that  $q(i-1, j-1) \not\models \phi_{j-1}(H)$ .

Hence,  $\mathcal{Q}(j-1)$  holds. In fact, for all  $l+1 \geq j \geq i$ ,  $\mathcal{Q}(j-1)$  holds and therefore  $\mathcal{P}(i-1)$  holds. In fact,  $\mathcal{P}(i)$  holds for all  $0 \leq i \leq k+1$ . The lemma is then given by  $\mathcal{P}(0)$  applied with  $j=1$ .  $\square$

The proof of Lemma 73 is now direct.

*Proof.* Assume that  $(l, C, k)$  is a positive instance of the hitting set problem Hit. Consider a hitting set  $H \subseteq [1, \dots, l]$  with  $|H| \leq k$ . We let  $\phi := \phi(l, H) \in \text{CTL}(\{p, \bar{p}\}, \mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$  since  $\mathbf{X} \in \mathbf{U}^t$ . We have  $|\phi| = l+1$ . By Lemma 75,  $\varphi$  accepts  $K_l^{\mathcal{P}}$  and rejects both  $K_l^{\mathcal{N}}$ . By Lemma 78,  $\phi$  rejects  $K_{\exists > k}^l$ . Consider now some  $1 \leq i \leq n$ . Since  $H$  is a hitting set, we have  $H \cap C_i \neq \emptyset$ . Hence, by Lemma 77,  $\phi(l, H)$  accepts the Kripke structure  $K_{(l, C_i)}$ . It follows that the CTL-formula  $\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . Hence,  $\text{In}_{(l, C, k)}^{\text{CTL}(\mathbf{X}), 0}$  is a positive instance of the  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, n)$  decision problem.

Assume now that  $\text{In}_{(l, C, k)}^{\text{CTL}(\mathbf{X}), 0}$  is a positive instance of the decision problem  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$ . Consider a CTL-formula  $\phi \in \text{CTL}(\{p, \bar{p}\}, \mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$  of size at most  $l+1$  that

accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . Since  $\phi$  accepts  $K_i^{\mathcal{P}} \in \mathcal{P}$  and rejects  $K_i^{\mathcal{N}} \in \mathcal{N}$ , it follows by Lemma 75 that  $\phi$  is an  $\mathbf{X}^l$ -formula. Let  $H \subseteq [1, \dots, l]$  be such that  $\phi = \phi(l, H)$ . Since  $\phi$  rejects  $K_{\exists > k}^l \in \mathcal{N}$ , it follows, by Lemma 78, that  $|H| \leq k$ . Consider some  $1 \leq i \leq n$ . Since  $\phi$  accepts  $K_{(l, C_i)} \in \mathcal{P}$ , it follows, by Lemma 77, that  $H \cap C_i \neq \emptyset$ . Therefore,  $H$  is a hitting set and  $(l, C, k)$  is a positive instance of the hitting set problem Hit.

We conclude with Corollary 76.  $\square$

Theorem 68 follows.

*Proof.* This is direct consequence of Lemma 73, Corollary 76, of the fact that the instance  $\ln_{(l, C, k)}^{\text{CTL}(\mathbf{X}), 0}$  can be computed in logarithmic space from  $(l, C, k)$ , and of Theorem 65.  $\square$

### 4.3.2 Without the next operator $\mathbf{X}$

In the previous subsection, we have seen that the CTL learning problem with a bounded amount of binary operator is NP-complete. However, as can be seen, the proof of NP-hardness heavily relies on the use of the operator  $\mathbf{X}$ . In this subsection, we focus on the CTL learning problem where the operator  $\mathbf{X}$  is not allowed anymore. We show that this decision problem is in NL, as stated in the theorem below.

**Theorem 79.** *For all  $U^t \subseteq \{\mathbf{F}, \mathbf{G}, \neg\}$ ,  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ , the  $\text{CTL}_{\text{Learn}}(U^t, \emptyset, B^l, n)$  decision problem is in NL. If  $\mathbf{F} \in U^t$  or  $\mathbf{G} \in U^t$ , then the  $\text{CTL}_{\text{Learn}}(U^t, \emptyset, B^l, n)$  decision problem is NL-complete.*

Proving that the decision problem is NL-hard is rather straightforward and will be handled as a second step. Let us first show that this decision problem is in NL.

To prove this, we are going to proceed similarly to the LTL case (except that this case is more involved), i.e. we first consider CTL-formulas that do not use binary operators at all, and then consider the case of CTL-formulas using binary operators.

Let us first tackle the case of CTL-formulas using no binary operators. Our goal is to successively restrict the set of CTL-formulas that is sufficient to consider. More precisely, we show that sequentially using several operators in a row is useless. We start by showing that using twice in a row either of the operators  $\mathbf{F}$  or  $\mathbf{G}$  is useless. The formal statement that we give below is in the context of ATL-formulas, more general than CTL-formulas, because we will use this statement later on in this paper.

**Lemma 80.** *Consider some  $k \in \mathbb{N}_1$ ,  $A \subseteq A' \subseteq [1, \dots, k]$ . Let  $\phi$  be any ATL-formula with  $k$  agents. We have:*

$$\langle\langle A \rangle\rangle \mathbf{G} \langle\langle A' \rangle\rangle \mathbf{G} \phi \equiv \langle\langle A' \rangle\rangle \mathbf{G} \langle\langle A \rangle\rangle \mathbf{G} \phi \equiv \langle\langle A \rangle\rangle \mathbf{G} \phi$$

and dually

$$\langle\langle A \rangle\rangle \mathbf{F} \langle\langle A' \rangle\rangle \mathbf{F} \phi \equiv \langle\langle A' \rangle\rangle \mathbf{F} \langle\langle A \rangle\rangle \mathbf{F} \phi \equiv \langle\langle A' \rangle\rangle \mathbf{F} \phi$$

*Proof.* We start with the operator  $\mathbf{G}$ .

- By definition of the globally operator  $\mathbf{G}$ , we have  $\langle\langle A' \rangle\rangle \mathbf{G} \phi \implies \phi$ , hence  $\langle\langle A \rangle\rangle \mathbf{G} \langle\langle A' \rangle\rangle \mathbf{G} \phi \implies \langle\langle A \rangle\rangle \mathbf{G} \phi$ . Also, we have  $\langle\langle A' \rangle\rangle \mathbf{G} \langle\langle A \rangle\rangle \mathbf{G} \phi \implies \langle\langle A \rangle\rangle \mathbf{G} \phi$ .
- Let us show that  $\langle\langle A \rangle\rangle \mathbf{G} \phi \implies \langle\langle A \rangle\rangle \mathbf{G} \langle\langle A \rangle\rangle \mathbf{G} \phi$ . Consider a concurrent game structure  $C$  and any state  $q \in Q$ . Assume that  $q \models \langle\langle A \rangle\rangle \mathbf{G} \phi$ . Consider a strategy profile  $s^A \in \mathcal{S}_A$  for the coalition of agents  $A$  such that, for all  $\rho \in \text{Out}^Q(q, s^A)$ , we have  $\rho \models \mathbf{G} \phi$ . We claim that we have for all  $\rho \in \text{Out}^Q(q, s^A)$ ,  $\rho \models \mathbf{G} \langle\langle A \rangle\rangle \mathbf{G} \phi$ . Indeed, consider any  $\rho \in$



$\text{Out}^Q(q, s^A)$  and  $i \in \mathbb{N}_1$ . Let  $\tilde{s}^A$  be a strategy profile for the coalition  $A$  that coincides with  $s^A$  after  $\rho[: i - 1]$ . That is, for all  $a \in A$  and  $\theta \in Q^+$ , we have  $\tilde{s}_a^A(\theta) := s_a^A(\rho[: i - 1] \cdot \theta)$ . Consider then any  $\theta \in \text{Out}^Q(\rho[: i], \tilde{s}^A)$ . By definition of the strategy profiles  $\tilde{s}^A$  and  $s^A$ , we have that  $\rho[: i - 1] \cdot \theta \in \text{Out}^Q(q, s^A)$ . Therefore, for all  $j \in \mathbb{N}_1$ , we have  $(\rho[: i - 1] \cdot \theta)[j] \models \phi$ . Thus, for all  $j \in \mathbb{N}_1$ , we have  $\theta[j] \models \phi$ . That is,  $\theta \models \mathbf{G} \phi$ . In fact,  $\rho[: i] \models \langle\langle A \rangle\rangle \mathbf{G} \phi$ , which holds for all  $i \in \mathbb{N}_1$ . Thus,  $\rho \models \mathbf{G} \langle\langle A \rangle\rangle \mathbf{G} \phi$ . We obtain that  $\langle\langle A \rangle\rangle \mathbf{G} \phi \implies \langle\langle A \rangle\rangle \mathbf{G} \langle\langle A \rangle\rangle \mathbf{G} \phi$ .

Since  $A \subseteq A'$ , we have, for all ATL-formulas  $\phi'$ ,  $\langle\langle A \rangle\rangle \mathbf{G} \phi' \implies \langle\langle A' \rangle\rangle \mathbf{G} \phi'$ . We can conclude that  $\langle\langle A \rangle\rangle \mathbf{G} \phi \implies \langle\langle A' \rangle\rangle \mathbf{G} \langle\langle A \rangle\rangle \mathbf{G} \phi$  and  $\langle\langle A \rangle\rangle \mathbf{G} \phi \implies \langle\langle A \rangle\rangle \mathbf{G} \langle\langle A' \rangle\rangle \mathbf{G} \phi$ .

We now turn to the operator  $\mathbf{F}$ , with dual arguments.

- By definition of the eventually operator  $\mathbf{F}$ , we have  $\phi \implies \langle\langle A \rangle\rangle \mathbf{F} \phi$ , hence  $\langle\langle A' \rangle\rangle \mathbf{F} \phi \implies \langle\langle A' \rangle\rangle \mathbf{F} \langle\langle A \rangle\rangle \mathbf{F} \phi$ . Also, we have  $\langle\langle A' \rangle\rangle \mathbf{F} \phi \implies \langle\langle A \rangle\rangle \mathbf{F} \langle\langle A' \rangle\rangle \mathbf{F} \phi$ .
- Consider now two coalitions  $A_1, A_2 \subseteq A'$ , a concurrent game structure  $C$  and any state  $q \in Q$ . Assume that  $q \models \langle\langle A_1 \rangle\rangle \mathbf{F} \langle\langle A_2 \rangle\rangle \mathbf{F} \phi$ . Let us show that  $q \models \langle\langle A' \rangle\rangle \mathbf{F} \phi$ . Consider a strategy profile  $s^{A_1} \in \mathbf{S}_{A_1}$  for the coalition of agents  $A_1$  such that, for all  $\rho \in \text{Out}^Q(q, s^{A_1})$ , we have  $\rho \models \mathbf{F} \langle\langle A_2 \rangle\rangle \mathbf{F} \phi$ . We let  $\text{Already}(\langle\langle A_2 \rangle\rangle \mathbf{F} \phi) := \{\rho \in Q^+ \cup Q^\omega \mid \exists i_\rho \in \mathbb{N}_1, i_\rho \leq |\rho|, \rho[: i_\rho] \models \langle\langle A_2 \rangle\rangle \mathbf{F} \phi\}$ . By definition of the strategy  $s^{A_1}$ , we have  $\text{Out}^Q(q, s^{A_1}) \subseteq \text{Already}(\langle\langle A_2 \rangle\rangle \mathbf{F} \phi)$ . Then, for all such  $\rho \in \text{Already}(\langle\langle A_2 \rangle\rangle \mathbf{F} \phi)$ , we let  $i_\rho \in \mathbb{N}_1$  be the least index such that  $\rho[: i_\rho] \models \langle\langle A_2 \rangle\rangle \mathbf{F} \phi$  and  $\theta_\rho \in Q^* \cup Q^\omega$  be such that  $\rho = \rho[: i_\rho - 1] \cdot \theta_\rho$ . We also let  $s^{\rho, A_2} \in \mathbf{S}_{A_2}$  be a strategy profile for the coalition  $A_2$  such that for all  $\theta \in \text{Out}^Q(\rho[: i_\rho], s^{\rho, A_2})$ , we have  $\theta \models \mathbf{F} \phi$ .

We now define the strategy profile  $s^{A'}$  such that, for all  $\rho \in Q^+$ :

$$s^{A'}(\rho) \text{ coincides with } \begin{cases} s^{A_1}(\rho) \text{ on } A_1 & \text{if } \rho \notin \text{Already}(\langle\langle A_2 \rangle\rangle \mathbf{F} \phi) \\ s^{\rho, A_2}(\theta_\rho) \text{ on } A_2 & \text{if } \rho \in \text{Already}(\langle\langle A_2 \rangle\rangle \mathbf{F} \phi) \end{cases}$$

where, for all coalitions of agents  $X, X_1, X_2$ , we say that a strategy  $s \in \mathbf{S}_{X_1}$  coincides with another strategy  $s' \in \mathbf{S}_{X_2}$  on  $X \subseteq X_1 \cap X_2$  if, for all  $a \in X$ , we have  $s_a = s'_a$ .

We claim that for all  $\rho \in \text{Out}^Q(q, s^{A'})$ ,  $\rho \models \mathbf{F} \phi$ . Indeed, consider any  $\rho \in \text{Out}^Q(q, s^{A'})$ . By definition, on finite paths not in  $\text{Already}(\langle\langle A_2 \rangle\rangle \mathbf{F} \phi)$ , the strategy  $s^{A'}$  coincides with the strategy  $s^{A_1}$  on  $A_1$ . Since  $\text{Out}^Q(q, s^{A_1}) \subseteq \text{Already}(\langle\langle A_2 \rangle\rangle \mathbf{F} \phi)$ , it follows that  $\rho \in \text{Already}(\langle\langle A_2 \rangle\rangle \mathbf{F} \phi)$ . Therefore, we have  $\rho = \rho[: i_\rho - 1] \cdot \theta_\rho$ . In addition, by definition of the strategy  $s^{A'}$ , we have  $\theta_\rho \in \text{Out}^Q(\rho[: i_\rho], s^{\rho, A_2})$ . By choice of the strategy  $s^{\rho, A_2}$ , this implies  $\theta_\rho \models \mathbf{F} \phi$ . It follows that  $\rho \models \mathbf{F} \phi$ . Therefore, for all  $\rho \in \text{Out}^Q(q, s^{A'})$ , we have  $\rho \models \mathbf{F} \phi$ . Thus,  $q \models \langle\langle A' \rangle\rangle \mathbf{F} \phi$ . Hence, we have proved that  $\langle\langle A_1 \rangle\rangle \mathbf{F} \langle\langle A_2 \rangle\rangle \mathbf{F} \phi \implies \langle\langle A' \rangle\rangle \mathbf{F} \phi$ .

We obtain the desired implications since  $A, A' \subseteq A'$ .

We obtain the equivalences for both operators  $\mathbf{F}$  and  $\mathbf{G}$ . □

Let us now come back to the more restrictive context of CTL-formulas where there are only two different strategic quantifiers:  $\langle\langle \emptyset \rangle\rangle$  (i.e.  $\forall$ ) and  $\langle\langle \{1\} \rangle\rangle$  (i.e.  $\exists$ ). To properly express the above lemma in this context, we define below the notion of dominating quantifiers (between  $\exists$  and  $\forall$ ) used with the operators  $\mathbf{F}$  and  $\mathbf{G}$ .

**Definition 81.** For all  $Q_1, Q_2 \in \{\exists, \forall\}$ , we define  $\text{Dom}_{\mathbf{F}}(Q_1, Q_2) \in \{Q_1, Q_2\}$  and  $\text{Dom}_{\mathbf{G}}(Q_1, Q_2) \in \{Q_1, Q_2\}$  as follows:

$$\text{Dom}_{\mathbf{F}}(Q_1, Q_2) := \begin{cases} \exists & \text{if } Q_1 = \exists \text{ or } Q_2 = \exists \\ \forall & \text{otherwise} \end{cases}$$

and

$$\text{Dom}_{\mathbf{G}}(Q_1, Q_2) := \begin{cases} \forall & \text{if } Q_1 = \forall \text{ or } Q_2 = \forall \\ \exists & \text{otherwise} \end{cases}$$

Note that  $\text{Dom}_{\mathbf{G}}$  is the dual of the operator  $\text{Dom}_{\mathbf{F}}$ , as stated in the observation below.

**Observation 82.** For all CTL-formula  $\phi$  and  $Q, Q_1, Q_2 \in \{\exists, \forall\}$ , we have:

$$\neg \text{Dom}_{\mathbf{F}}(Q_1, Q_2) \mathbf{F}\phi \equiv \text{Dom}_{\mathbf{G}}(\neg Q_1, \neg Q_2) \mathbf{G} \neg\phi \quad (1)$$

$$\phi \implies Q \mathbf{F}\phi \quad (2)$$

$$Q \mathbf{G}\phi \implies \phi \quad (3)$$

where  $\neg\forall := \exists$  and  $\neg\exists := \forall$ .

*Proof.* The equivalence (1) is a straightforward consequence of the following equivalences, for  $\phi'$  any CTL-formula:  $\neg\forall \mathbf{G}\phi' \equiv \exists \mathbf{F}\neg\phi'$ ,  $\neg\exists \mathbf{F}\phi' \equiv \forall \mathbf{G}\neg\phi'$ ,  $\neg\exists \mathbf{G}\phi' \equiv \forall \mathbf{F}\neg\phi'$  and  $\neg\forall \mathbf{F}\phi' \equiv \exists \mathbf{G}\neg\phi'$ . The implications (2) and (3) come from the definition of the operators  $\mathbf{F}$  and  $\mathbf{G}$  and the fact that, in all Kripke structures  $K$ , for all states  $q \in Q$  and for all  $\rho \in \text{Out}^Q(q)$ , we have  $\rho[1] = q$ .  $\square$

We can now state below the corollary of Lemma 80 with CTL-formulas. (Which justifies the terminology defined above of *dominating* quantifiers.)

**Corollary 83.** Let  $\phi$  be any CTL-formula and  $Q_1, Q_2 \in \{\exists, \forall\}$ . We have:

$$Q_1 \mathbf{G} Q_2 \mathbf{G}\phi \equiv Q_2 \mathbf{G} Q_1 \mathbf{G}\phi \equiv \text{Dom}_{\mathbf{G}}(Q_1, Q_2) \mathbf{G}\phi$$

and dually

$$Q_1 \mathbf{F} Q_2 \mathbf{F}\phi \equiv Q_2 \mathbf{F} Q_1 \mathbf{F}\phi \equiv \text{Dom}_{\mathbf{F}}(Q_1, Q_2) \mathbf{F}\phi$$

*Proof.* This is a direct consequence of Lemma 80 and the fact that  $\forall$  stands for  $\langle\langle\emptyset\rangle\rangle$  and  $\exists$  stands for  $\langle\langle\{1\}\rangle\rangle$ .  $\square$

We deduce the corollary below stating equivalences over CTL-formulas alternating the  $\mathbf{F}$  and  $\mathbf{G}$  operators.

**Corollary 84.** Let  $\phi$  be any CTL-formula and  $Q_1, Q_2, Q_3, Q_4 \in \{\exists, \forall\}$ . Let  $Q_{\mathbf{F}} := \text{Dom}_{\mathbf{F}}(Q_1, Q_3)$  and  $Q_{\mathbf{G}} := \text{Dom}_{\mathbf{G}}(Q_2, Q_4)$ . Assume that  $Q_1 = Q_{\mathbf{F}}$  and  $Q_4 = Q_{\mathbf{G}}$ . Then:

$$Q_1 \mathbf{F} Q_2 \mathbf{G} Q_3 \mathbf{F} Q_4 \mathbf{G}\phi \equiv Q_{\mathbf{F}} \mathbf{F} Q_{\mathbf{G}} \mathbf{G}\phi$$

Dually, letting  $Q_{\mathbf{G}} := \text{Dom}_{\mathbf{G}}(Q_1, Q_3)$  and  $Q_{\mathbf{F}} := \text{Dom}_{\mathbf{F}}(Q_2, Q_4)$ , if  $Q_1 = Q_{\mathbf{G}}$  and  $Q_4 = Q_{\mathbf{F}}$ , then:

$$Q_1 \mathbf{G} Q_2 \mathbf{F} Q_3 \mathbf{G} Q_4 \mathbf{F}\phi \equiv Q_{\mathbf{G}} \mathbf{G} Q_{\mathbf{F}} \mathbf{F}\phi$$

*Proof.* We prove the first equivalence, the second is then given by Observation 82 (Eq. (1)).

By Corollary 83 and Observation 82 (Eq. (2)), we have:

$$Q_{\mathbf{G}} \mathbf{G}\phi \equiv Q_2 \mathbf{G} Q_4 \mathbf{G}\phi \implies Q_2 \mathbf{G} Q_3 \mathbf{F} Q_4 \mathbf{G}\phi$$

Since  $Q_{\mathbf{F}} = Q_1$ , we have:

$$Q_{\mathbf{F}} \mathbf{F} Q_{\mathbf{G}} \mathbf{G}\phi \implies Q_1 \mathbf{F} Q_2 \mathbf{G} Q_3 \mathbf{F} Q_4 \mathbf{G}\phi$$

On the other hand, we have by Observation 82 (Eq. (3)):

$$Q_2 \mathbf{G} Q_3 \mathbf{F} Q_4 \mathbf{G} \phi \implies Q_3 \mathbf{F} Q_4 \mathbf{G} \phi$$

Therefore, by Corollary 83 and since  $Q_{\mathbf{G}} = Q_4$ :

$$Q_1 \mathbf{F} Q_2 \mathbf{G} Q_3 \mathbf{F} Q_4 \mathbf{G} \phi \implies Q_1 \mathbf{F} Q_3 \mathbf{F} Q_4 \mathbf{G} \phi \equiv Q_{\mathbf{F}} \mathbf{F} Q_4 \mathbf{G} \phi \implies Q_{\mathbf{F}} \mathbf{F} Q_{\mathbf{G}} \mathbf{G} \phi$$

We obtain the desired equivalence.  $\square$

We deduce that it is useless to use quantifiers between  $\exists \mathbf{F}$  and  $\forall \mathbf{G}$ .

**Lemma 85.** *Let  $\phi$  be any CTL-formula and  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$  be a sequence of quantifiers. We have:*

$$\exists \mathbf{F} \mathbf{Qt} \forall \mathbf{G} \phi \equiv \exists \mathbf{F} \forall \mathbf{G} \phi$$

and

$$\forall \mathbf{G} \mathbf{Qt} \exists \mathbf{F} \phi \equiv \forall \mathbf{G} \exists \mathbf{F} \phi$$

*Proof.* We prove the result by induction on the size of  $\mathbf{Qt}$ . If  $\mathbf{Qt}$  is the empty sequence, both equivalences are straightforward. Assume now that both equivalences hold for all sequences  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$  of size at most  $k$ , for some  $k \in \mathbb{N}$ . Consider a sequence of quantifiers  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^{k+1}$ . Let us consider the sequence  $\exists \mathbf{F} \mathbf{Qt} \forall \mathbf{G}$ , the arguments are similar for the other one. If  $k+1$  is odd, then the sequence  $\exists \mathbf{F} \mathbf{Qt} \forall \mathbf{G}$  features an operator  $\mathbf{F}$  or  $\mathbf{G}$  used twice in a row. By Corollary 83, there is some  $\mathbf{Qt}' \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$  with  $|\mathbf{Qt}'| \leq k$  such that  $\exists \mathbf{F} \mathbf{Qt} \forall \mathbf{G} \phi \equiv \exists \mathbf{F} \mathbf{Qt}' \forall \mathbf{G} \phi$ . We can then apply our induction hypothesis.

Assume now that  $k+1$  is even and that the sequence  $\exists \mathbf{F} \mathbf{Qt} \forall \mathbf{G}$  does not feature an operator  $\mathbf{F}$  or  $\mathbf{G}$  used twice in a row. If  $|\mathbf{Qt}| = 2$ , we can apply Corollary 84. Assume now that  $|\mathbf{Qt}| \geq 4$ . Let us write  $\mathbf{Qt}$  as  $\mathbf{Qt} = \mathbf{T}_1 \cdots \mathbf{T}_{2n}$  where  $\mathbf{T}_i \in \{\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G}\}$  for all  $1 \leq i \leq 2n$ . By assumption, for all even  $1 \leq i \leq 2n$ ,  $\mathbf{T}_i$  is an  $\mathbf{F}$  operator, whereas for all odd  $1 \leq i \leq 2n$ ,  $\mathbf{T}_i$  is an  $\mathbf{G}$  operator. We denote by  $Q_i$  the  $\exists$  or  $\forall$  quantifier associated with  $\mathbf{T}_i$ . There are three cases:

- Assume that  $Q_1 = \exists$ . Then, we necessarily have  $\text{Dom}_{\mathbf{G}}(Q_1, Q_3) = Q_3$ , and  $\text{Dom}_{\mathbf{F}}(\exists, Q_2) = \exists$ . Therefore, by Corollary 84, we have  $\exists \mathbf{F} \mathbf{Qt} \forall \mathbf{G} \phi \equiv \exists \mathbf{F} \mathbf{T}_3 \cdots \mathbf{T}_{2n} \forall \mathbf{G} \phi$ . We can then apply our induction hypothesis.
- Similarly, assume that  $Q_{2n} = \forall$ . Then, we necessarily have  $\text{Dom}_{\mathbf{F}}(Q_{2(n-1)}, Q_{2n}) = Q_{2(n-1)}$ , and  $\text{Dom}_{\mathbf{G}}(Q_{2n-1}, \forall) = \forall$ . Therefore, by Corollary 84, we have  $\exists \mathbf{F} \mathbf{Qt} \forall \mathbf{G} \phi \equiv \exists \mathbf{F} \mathbf{T}_1 \cdots \mathbf{T}_{2(n-1)} \forall \mathbf{G} \phi$ . We can then apply our induction hypothesis.
- Otherwise, we have  $\mathbf{T}_1 = \forall \mathbf{G}$  and  $\mathbf{T}_{2n} = \exists \mathbf{F}$ . Therefore, by our induction hypothesis and Corollary 84,  $\exists \mathbf{F} \mathbf{Qt} \forall \mathbf{G} \phi = \exists \mathbf{F} \mathbf{T}_1 \cdots \mathbf{T}_{2n} \forall \mathbf{G} \phi \equiv \exists \mathbf{F} \forall \mathbf{G} \exists \mathbf{F} \forall \mathbf{G} \phi \equiv \exists \mathbf{F} \forall \mathbf{G} \phi$ .

Thus the equivalence holds for all sequences of operators  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$  of size  $k+1$ . The lemma follows.  $\square$

We also deduce that it is useless to use too long sequences of operators next to  $\exists \mathbf{F} \forall \mathbf{G}$ .

**Lemma 86.** *Let  $\phi$  be any CTL-formula. For all  $\mathbf{Qt}, \mathbf{Qt}' \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$ , there is  $\mathbf{Qt}_s, \mathbf{Qt}'_s \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$  of size at most 3 such that:*

$$\mathbf{Qt} \exists \mathbf{F} \forall \mathbf{G} \mathbf{Qt}' \phi \equiv \mathbf{Qt}_s \exists \mathbf{F} \forall \mathbf{G} \mathbf{Qt}'_s \phi$$

Similarly, for all  $\mathbf{Qt}, \mathbf{Qt}' \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$ , there is  $\mathbf{Qt}_s, \mathbf{Qt}'_s \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$  of size at most 3 such that:

$$\mathbf{Qt} \forall \mathbf{G} \exists \mathbf{F} \mathbf{Qt}' \phi \equiv \mathbf{Qt}_s \forall \mathbf{G} \exists \mathbf{F} \mathbf{Qt}'_s \phi$$

*Proof.* We prove the result for  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$  with the sequence  $\exists \mathbf{F} \forall \mathbf{G}$ , the three other cases are analogous.

We prove by induction on the size of  $\mathbf{Qt}$  that there is  $\mathbf{Qt}_s \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$  of size at most 3 such that:  $\mathbf{Qt} \exists \mathbf{F} \forall \mathbf{G} \phi \equiv \mathbf{Qt}_s \exists \mathbf{F} \forall \mathbf{G} \phi$ . This obviously holds if  $\mathbf{Qt}$  is of size at most 3. Assume now that it holds for all sizes  $i \leq k$  for some  $k \geq 3$ . Consider a sequence  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^{k+1}$ . We let  $\mathbf{Qt}$  be equal to  $\mathbf{Qt} = \mathbf{Qt}' \cdot \mathsf{T}_1 \cdot \mathsf{T}_2 \cdot \mathsf{T}_3 \cdot \mathsf{T}_4$  where, for all  $1 \leq i \leq n$ , we have  $\mathsf{T}_i \in \{\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G}\}$ . We assume that in the sequence  $\mathbf{Qt} \exists \mathbf{F} \forall \mathbf{G}$ , no  $\mathbf{F}$  or  $\mathbf{G}$  operator appears twice in a row, otherwise we can apply Corollary 83 and our induction hypothesis. For  $1 \leq i \leq 4$ , we let  $Q_i$  denote the  $\exists$  or  $\forall$  quantifier associated with the operator  $\mathsf{T}_i$ . There are two cases:

- If  $\mathsf{T}_2 = \forall \mathbf{G}$  or  $\mathsf{T}_3 = \exists \mathbf{F}$ , Corollary 84 gives that  $\mathbf{Qt} \exists \mathbf{F} \forall \mathbf{G} \phi \equiv \mathbf{Qt}' \cdot \mathsf{T}_1 \cdot \mathsf{T}_2 \cdot \exists \mathbf{F} \forall \mathbf{G} \phi$ . We can then apply our induction hypothesis to  $\mathbf{Qt}' \cdot \mathsf{T}_1 \cdot \mathsf{T}_2 \cdot \exists \mathbf{F} \forall \mathbf{G} \phi$  since  $|\mathbf{Qt}' \cdot \mathsf{T}_1 \cdot \mathsf{T}_2| \leq k$ .
- Otherwise, we have both  $\text{Dom}_{\mathbf{F}}(Q_1, Q_3) = Q_1$  and  $\text{Dom}_{\mathbf{G}}(Q_2, Q_4) = Q_4$ . Therefore, by Corollary 84, we have  $\mathbf{Qt} \exists \mathbf{F} \forall \mathbf{G} \phi \equiv \mathbf{Qt}' \cdot \mathsf{T}_1 \cdot \mathsf{T}_4 \cdot \exists \mathbf{F} \forall \mathbf{G} \phi$ . We can then apply our induction hypothesis to  $\mathbf{Qt}' \cdot \mathsf{T}_1 \cdot \mathsf{T}_4 \exists \mathbf{F} \forall \mathbf{G} \phi$  since  $|\mathbf{Qt}' \cdot \mathsf{T}_1 \cdot \mathsf{T}_4| \leq k$ .

Overall, the equivalences also holds for all sequences  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^{k+1}$ . The lemma follows.  $\square$

Finally, let us consider the case where at most one of the two operators  $\exists \mathbf{F}$  or  $\forall \mathbf{G}$  is used.

**Lemma 87.** *Let  $\phi$  be any CTL-formula. For all  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G})^*$ , there is  $\mathbf{Qt}_s \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G})^*$  of size at most 5 such that  $\mathbf{Qt} \phi \equiv \mathbf{Qt}_s \phi$ .*

*Similarly, for all  $\mathbf{Qt} \in (\exists \mathbf{G}, \forall \mathbf{G}, \exists \mathbf{F})^*$ , there is  $\mathbf{Qt}_s \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G})^*$  of size at most 5 such that  $\mathbf{Qt} \phi \equiv \mathbf{Qt}_s \phi$ .*

*Proof.* We prove the result for the first case, the second one is analogous. We prove the result by induction on the size of  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G})^*$  that there is  $\mathbf{Qt}_s \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G})^*$  of size at most 5 such that  $\mathbf{Qt} \phi \equiv \mathbf{Qt}_s \phi$ . This obviously holds if  $\mathbf{Qt}$  is of size at most 5. Assume now that it holds for all sizes  $i \leq k$  of  $\mathbf{Qt}$  for some  $k \geq 5$ . Consider a sequence  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G})^{k+1}$ . We denote  $\mathbf{Qt}$  as follows  $\mathbf{Qt} = \mathsf{T}_1 \cdot \mathsf{T}_2 \cdot \mathsf{T}_3 \cdot \mathsf{T}_4 \cdot \mathsf{T}_5 \cdot \mathsf{T}_6 \cdot \mathbf{Qt}'$ . We assume that in  $\mathbf{Qt}$ , no  $\mathbf{F}$  or  $\mathbf{G}$  operator appears twice in a row, otherwise we can apply Corollary 83 and our induction hypothesis. For  $1 \leq i \leq 6$ , we let  $Q_i$  denote the  $\exists$  or  $\forall$  quantifier associated with the operator  $\mathsf{T}_i$ . There are two cases.

- Assume that there is some  $1 \leq i \leq 6$  such that  $\mathsf{T}_i = \exists \mathbf{F}$ . If  $i \leq 3$ , then we have  $\text{Dom}_{\mathbf{F}}(Q_i, Q_{i+2}) = Q_i$  and  $\text{Dom}_{\mathbf{G}}(Q_{i+1}, Q_{i+3}) = Q_{i+3}$  since  $Q_{i+1} = Q_{i+3} = \exists$ . Thus, by Corollary 84, for all CTL-formulas  $\phi'$ , we have  $\mathsf{T}_i \cdot \mathsf{T}_{i+1} \cdot \mathsf{T}_{i+2} \cdot \mathsf{T}_{i+3} \cdot \phi' \equiv \mathsf{T}_i \cdot \mathsf{T}_{i+3} \cdot \phi'$ . We can then apply our induction hypothesis to conclude. If  $i \geq 4$ , then we have  $\text{Dom}_{\mathbf{F}}(Q_{i-2}, Q_i) = Q_i$  and  $\text{Dom}_{\mathbf{G}}(Q_{i-3}, Q_{i-1}) = Q_{i-3}$  since  $Q_{i-3} = Q_{i-1} = \exists$ . Thus, by Corollary 84, for all CTL-formulas  $\phi'$ , we have  $\mathsf{T}_{i-3} \cdot \mathsf{T}_{i-2} \cdot \mathsf{T}_{i-1} \cdot \mathsf{T}_i \cdot \phi' \equiv \mathsf{T}_{i-3} \cdot \mathsf{T}_i \cdot \phi'$ . We can then apply our induction hypothesis to conclude.
- Otherwise, we have  $\mathbf{Qt} \in (\forall \mathbf{F}, \exists \mathbf{G})^{k+1}$  and thus we can apply Corollary 84 and our induction hypothesis to conclude.

Therefore, the result holds for sequences  $\mathbf{Qt} \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G})^{k+1}$ . The lemma follows.  $\square$

Overall, we obtain the lemma below, bounding the size of CTL-formulas (using only unary operators) that is sufficient to consider.

**Lemma 88.** *There is a bound  $B \in \mathbb{N}$  such that, for all  $U^t \subseteq \{\neg, \mathbf{F}, \mathbf{G}\}$ , for all CTL-formulas  $\phi = \mathbf{Qt} \cdot \phi' \in \text{CTL}(\text{Prop}, U^t, \emptyset, \emptyset, 0)$ , there is a sequence of CTL-operators  $\mathbf{Qt}'$  such that  $\phi'' = \mathbf{Qt}' \cdot \phi' \in \text{CTL}(\text{Prop}, U^t, \emptyset, \emptyset, 0)$ ,  $|\mathbf{Qt}| \leq B$ , and  $\phi \equiv \phi''$ .*

*Proof.* Let us first consider any  $\phi = \mathbf{Qt} \cdot \phi' \in \text{CTL}(\text{Prop}, U^t, \emptyset, \emptyset, 0)$  with  $\mathbf{Qt}$  as follows  $\mathbf{Qt} = \mathbf{T}_1 \cdots \mathbf{T}_n$  where, for all  $1 \leq i \leq n$ , we have  $\mathbf{T}_i \in \{\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G}\}$ . There are two cases:

- Assume that  $\phi$  uses both operators  $\exists \mathbf{F}$  and  $\forall \mathbf{G}$ . Consider  $1 \leq i < j \leq n$  such that  $\mathbf{T}_i, \mathbf{T}_j \in \{\exists \mathbf{F}, \forall \mathbf{G}\}$  and  $\mathbf{T}_i \neq \mathbf{T}_j$ . By Lemma 85, we have  $\phi \equiv \mathbf{T}_1 \cdots \mathbf{T}_i \cdot \mathbf{T}_j \cdots \mathbf{T}_n \phi'$ . Furthermore, by Lemma 86, there is  $\mathbf{Qt}_s, \mathbf{Qt}'_s \in (\exists \mathbf{F}, \forall \mathbf{F}, \exists \mathbf{G}, \forall \mathbf{G})^*$  of size at most 3 such that:  $\mathbf{T}_1 \cdots \mathbf{T}_i \cdot \mathbf{T}_j \cdots \mathbf{T}_n \phi' \equiv \mathbf{Qt}_s \cdot \mathbf{T}_i \cdot \mathbf{T}_j \cdot \mathbf{Qt}'_s \phi$ . We let  $\mathbf{Qt}' := \mathbf{Qt}_s \cdot \mathbf{T}_i \cdot \mathbf{T}_j \cdot \mathbf{Qt}'_s \phi' \in \text{CTL}(\text{Prop}, U^t, \emptyset, \emptyset)$ . We have  $|\mathbf{Qt}'| \leq 8$  and  $\mathbf{Qt}' \cdot \phi' \equiv \mathbf{Qt} \cdot \phi'$ .
- Assume  $\phi$  does not use both operators  $\exists \mathbf{F}$  or  $\forall \mathbf{G}$ . Then, by Lemma 87, there is  $\mathbf{Qt}'$  of size at most 5 such that  $\mathbf{Qt}' \cdot \phi' \equiv \mathbf{Qt} \cdot \phi'$ .

Thus, in both cases, there is a sequence of CTL-operators  $\mathbf{Qt}'$  of size at most 8 such that  $\mathbf{Qt}' \cdot \phi' \equiv \mathbf{Qt} \cdot \phi'$  with at most 8 quantifiers.

Then, it is straightforward to handle the cases where the sequence of quantifiers  $\mathbf{Qt}$  uses negations, since we have the equivalence, for all CTL-formulas  $\phi$ ,  $\mathbf{G} \phi \equiv \neg \mathbf{F} \neg \phi$  and  $\mathbf{F} \phi \equiv \neg \mathbf{G} \neg \phi$ .  $\square$

We obtain a similar statement with formulas that can use (a bounded amount of) binary operators.

**Lemma 89.** *For all  $U^t \subseteq \{\neg, \mathbf{F}, \mathbf{G}\}$ ,  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ , there is a bound  $B_n \in \mathbb{N}$  such that, for all CTL-formulas  $\phi \in \text{CTL}(\text{Prop}, U^t, \emptyset, B^l, n)$ , there is a CTL-formula  $\phi' \in \text{CTL}(\text{Prop}, U^t, \emptyset, B^l, n)$  such that  $|\phi'| \leq B_n$  and  $\phi' \equiv \phi$ .*

*Proof.* Let  $B$  denote the bound from Lemma 88. We proceed by induction on  $n$  (note that the bound  $B_n$  depends on  $n$ ). For the case  $n = 0$ , it suffices to consider  $B_0 := B + 1$ . Assume now that it holds for some  $n \in \mathbb{N}$ . We let  $B_{n+1} := 2B_n + B + 1 \geq B_n$ . Consider a formula  $\phi \in \text{CTL}(\text{Prop}, U^t, \emptyset, B^l, n+1) \setminus \text{CTL}(\text{Prop}, U^t, \emptyset, B^l, n)$ . This formula can be written as  $\phi = \mathbf{Qt} \cdot (\phi_1 \bullet \phi_2)$  where  $\mathbf{Qt}$  is a sequence of unary operators,  $\bullet \in B^l$  is a binary operator and  $\phi_1, \phi_2 \in \text{CTL}(\text{Prop}, U^t, \emptyset, B^l, n)$ . By our induction hypothesis, there are formulas  $\phi'_1, \phi'_2 \in \text{CTL}(\text{Prop}, U^t, \emptyset, B^l, n)$  such that  $|\phi'_1|, |\phi'_2| \leq B_n$  and  $\phi_1 \equiv \phi'_1, \phi_2 \equiv \phi'_2$ . In addition, by Lemma 88, there is a sequence of operators  $\mathbf{Qt}'$  of size at most  $B$  such that, for all CTL-formulas  $\phi'$ , we have  $\mathbf{Qt} \cdot \phi' \equiv \mathbf{Qt}' \cdot \phi'$  and  $\mathbf{Qt}' \cdot (\phi_1 \bullet \phi_2) \in \text{CTL}(\text{Prop}, U^t, \emptyset, B^l, n+1)$ . Overall, we have  $\mathbf{Qt}' \cdot (\phi'_1 \bullet \phi'_2) \in \text{CTL}(\text{Prop}, U^t, \emptyset, B^l, n+1)$  with  $|\mathbf{Qt}' \cdot (\phi'_1 \bullet \phi'_2)| \leq B_{n+1}$  and  $\mathbf{Qt}' \cdot (\phi'_1 \bullet \phi'_2) \equiv \mathbf{Qt} \cdot (\phi_1 \bullet \phi_2)$ . Hence, our inductive property holds also for  $n+1$ . The lemmas follows.  $\square$

We can establish that deciding the learning CTL decision problem without the operator  $\mathbf{X}$  can be done in non-deterministic logarithmic space.

**Lemma 90.** *For all  $U^t \subseteq \{\mathbf{F}, \mathbf{G}, \neg\}$ ,  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ , the problem  $\text{CTL}_{\text{Learn}}(U^t, \emptyset, B^l, n)$  can be decided in non-deterministic logarithmic space.*

*Proof.* By Immerman-Szelepcsényi's theorem [20], we have  $\text{NL} = \text{coNL}$ . In other words, any problem that can be decided by a logarithmic-space Turing machine using only existential (TM) states can also be decided by a logarithmic-space Turing machine using only universal (TM) states.

Now, consider the bound  $B_n$  from Lemma 89. We let  $\widetilde{\text{CTL}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  denote a set of CTL-formula structures, i.e. CTL-formulas where the propositions are left unspecified. Each one of these formula structure can be seen, for some  $k \in \mathbb{N}$ , as functions  $\text{Prop}^k \rightarrow \text{CTL}(\text{Prop}, \mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$ , for all sets of propositions  $\text{Prop}$ . The set  $\widetilde{\text{CTL}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  corresponds to the set of all CTL-formula structures such that, for all non-empty sets of propositions  $\text{Prop}$ , there are propositions in  $\text{Prop}$  specifying them such that the obtained formula is in  $\text{CTL}(\text{Prop}, \mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  and of size at most  $B_n$ . Note that the set  $\widetilde{\text{CTL}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  is finite. For all  $\tilde{\phi} \in \widetilde{\text{CTL}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$ , let us exhibit an NL-algorithm  $\text{SatKripke}_{\tilde{\phi}}$  that decides, given the propositions specifying the formula structure  $\tilde{\phi}$  and a state in a Kripke structure, whether that state satisfies the obtained CTL-formula.

First, checking that a state satisfies a proposition is straightforward. Assume now that we have designed an NL-algorithm  $\text{SatKripke}_{\tilde{\phi}}$  for a CTL-formula structure  $\tilde{\phi}$ . The case of the formula structure  $\tilde{\psi} = \neg\tilde{\phi}$  is straightforward since  $\text{NL} = \text{coNL}$ . Consider now the formula structure  $\tilde{\psi} = \exists \mathbf{F} \tilde{\phi}$ . Consider any propositions specifying  $\tilde{\psi}$  into a CTL-formula  $\psi$  (and therefore  $\tilde{\phi}$  into a CTL-formula  $\phi$ ). Checking that a state  $q$  satisfies  $\psi$  amounts to guessing a path from  $q$  of size at most  $|Q|$  and checking, with a logarithmic space Turing machine using only existential (TM) states, that it satisfies  $\phi$  (by calling the algorithm  $\text{SatKripke}_{\tilde{\phi}}$ ). This induces an NL-algorithm. Consider now the formula  $\tilde{\psi} = \forall \mathbf{F} \tilde{\phi}$ . Since  $\text{NL} = \text{coNL}$ , there is a coNL-algorithm  $\text{SatKripke}'_{\tilde{\phi}}$  for the formula structure  $\tilde{\phi}$ . As above, consider any propositions specifying  $\tilde{\psi}$  into a CTL-formula  $\psi$  (and therefore  $\tilde{\phi}$  into a CTL-formula  $\phi$ ). Then, checking that a state  $q$  satisfies the formula  $\psi$  amounts to exploring, with universal (TM) states, paths of lengths at most  $|Q|$  from  $q$  and check that we encounter a state that satisfies  $\phi$  (by calling  $\text{SatKripke}'_{\tilde{\phi}}$ ). This induces a coNL-algorithm, and therefore an NL-algorithm as well. The arguments are similar for the operators  $\exists \mathbf{G}$  and  $\forall \mathbf{G}$ . Furthermore, consider a formula structure  $\tilde{\psi} = \tilde{\phi}_1 \bullet \tilde{\phi}_2$  for some binary operator  $\bullet \in \mathbf{B}^l$ , and assume that we have designed NL-algorithms  $\text{SatKripke}_{\tilde{\phi}_i}, \text{SatKripke}_{\neg\tilde{\phi}_i}$  for the CTL-formula structures  $\tilde{\phi}_i, \neg\tilde{\phi}_i$ , for  $i \in \{1, 2\}$ . In that case, we may rewrite  $\bullet$  in disjunctive normal form (for instance  $x_1 \Leftrightarrow x_2 \equiv (x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2)$ ). Then, an NL-algorithm  $\text{SatKripke}_{\tilde{\psi}}$  could guess which clause to satisfy and run at most two of the algorithms  $\text{SatKripke}_{\tilde{\phi}_1}, \text{SatKripke}_{\tilde{\phi}_2}, \text{SatKripke}_{\neg\tilde{\phi}_1}, \text{SatKripke}_{\neg\tilde{\phi}_2}$ . That way, we obtain an NL-algorithm  $\text{SatKripke}_{\tilde{\psi}}$ .

Overall, we have an NL-algorithm for all CTL-formula structures (and their negations since  $\text{NL} = \text{coNL}$ ) in  $\widetilde{\text{CTL}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$ . Let us now design an NL-algorithm for the  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  decision problem. That algorithm could do the following, on an input  $(\text{Prop}, \mathcal{P}, \mathcal{N}, B)$ :

1. Loop over all CTL-formula structures  $\tilde{\phi}$  in  $\widetilde{\text{CTL}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$
2. Letting  $k$  be the number of unspecified propositions in  $\tilde{\phi}$ , loop over all tuples  $t$  in  $\text{Prop}^k$  for which the CTL-formula obtained from  $\tilde{\phi}$  with  $t$  has size at most  $\min(B_n, B)$
3. Loop over:
  - all starting states  $q$  of all positive structures and run the NL-algorithm  $\text{SatKripke}_{\tilde{\phi}}$  on  $t$  and  $q$
  - all negative structures  $K \in \mathcal{N}$ , guess a starting state  $q$  in  $K$  and run the NL-algorithm  $\text{SatKripke}_{\neg\tilde{\phi}}$  on  $t$  and  $q$

Accept if all calls return positive answers

If the algorithm does not accept the input, then it rejects it. The first loop is entered a bounded number of times (independent of the input). Since of formula structures in  $\widetilde{\text{CTL}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  use

at most  $n$  occurrences of binary operators, the number  $k$  of unspecified variables in  $\tilde{\phi}$  is at most  $2^n$ , thus the second loop is entered at most  $|\text{Prop}|^{2^n}$  times, which is polynomial in  $|\text{Prop}|$ . Therefore, the algorithm that we have designed above runs in non-deterministic logarithmic space. Furthermore, it decides the problem  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  by Lemma 89.  $\square$

**NL-hardness** To establish NL-hardness, we are going to exhibit a reduction from the problem of reachability in a graph. We introduce in the definition below that decision problem as a sub-case of a more general decision problem of reachability in a two-player game, that we will use in the next section.

**Definition 91** (Reachability Game). *We denote by Reach the following decision problem:*

- *Input: two propositions  $p, \bar{p}$  and a  $\{1, 2\}$ -game structure  $G$  on  $\{p, \bar{p}\}$ , such that all state in  $G$  satisfy exactly one of the two propositions  $p$  and  $\bar{p}$ , and with a single starting state;*
- *Output: yes iff  $q \models \langle\langle\{1\}\rangle\rangle \mathbf{F} p$ .*

*Similarly, we consider the decision problem 1-Reach, where the game structure taken as input is in fact a  $\{1\}$ -game structure, i.e. it is a Kripke structure.*

**Theorem 92** ([19],[27]). *The decision problem Reach is P-complete and the decision problem 1-Reach is NL-complete under logarithmic space reductions.*

*Proof.* The decision problem Reach is equivalent to solving two-player reachability games, which is P-complete [19]. The decision problem 1-Reach is equivalent to solving reachability in a graph (“s-t connectivity”), which is NL-complete [27, Theorem 16.2].  $\square$

Let us define the reduction that we consider.

**Definition 93.** *Consider any input  $p, \bar{p}, K$  of the decision problem 1-Reach. We let  $K_{\bar{p}}$  be a single-state Kripke structure whose only state is labeled by  $\{\bar{p}\}$ .*

*We define the inputs  $\text{In}_{(p, \bar{p}, K)}^{\text{CTL}, \mathbf{F}} := (\{p, \bar{p}\}, \{K\}, \{K_{\bar{p}}\}, 2)$  and  $\text{In}_{(p, \bar{p}, K)}^{\text{CTL}, \mathbf{G}} := (\{p, \bar{p}\}, \{K_{\bar{p}}\}, \{K\}, 2)$  of a CTL learning problem.*

The definition above satisfies the lemma below.

**Lemma 94.** *Consider some set of unary operators  $\mathbf{U}^t \subseteq \{\mathbf{X}, \mathbf{F}, \mathbf{G}, \neg\}$ ,  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$ ,  $n \in \mathbb{N}$  and an input  $(p, \bar{p}, K)$  of the 1-Reach decision problem. Let  $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$ . If  $\mathbf{H} \in \mathbf{U}^t$ , then the input  $(p, \bar{p}, K)$  is a positive instance of the 1-Reach decision problem if and only if  $\text{In}_{(p, \bar{p}, K)}^{\text{CTL}, \mathbf{H}}$  is a positive instance of the  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  decision problem.*

*Proof.* Let us first consider the case  $\mathbf{H} = \mathbf{F}$ . Assume that  $(p, \bar{p}, K)$  is a positive instance of 1-Reach. We let  $\varphi := \exists \mathbf{F} p$ . We have  $|\varphi| = 2$ . Furthermore,  $K_{\bar{p}} \not\models \varphi$  and, by assumption,  $K \models \varphi$ . Hence,  $\text{In}_{(p, \bar{p}, K)}^{\text{CTL}, \mathbf{F}}$  is a positive instance of the  $\text{CTL}_{\text{Learn}}^2(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  decision problem.

On the other hand, assume that  $\text{In}_{(p, \bar{p}, K)}^{\text{CTL}, \mathbf{F}}$  is a positive instance of the  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  decision problem. Consider a separating formula  $\varphi$  of size at most 2 that accepts the Kripke structure  $K$  and rejects the Kripke structure  $K_{\bar{p}}$ . The structures  $K$  and  $K_{\bar{p}}$  satisfy the following property: for all states  $q$ ,  $q \models p$  if and only if  $q \models \neg \bar{p}$ . Let us show that on these structures, we have  $\varphi \implies \exists \mathbf{F} p$ . If  $\varphi$  uses a negation, then  $\varphi = \neg \bar{p}$  and  $\varphi \equiv p \implies \exists \mathbf{F} p$  on such structures. If  $\varphi$  uses a binary operator, then it is equivalent to either  $p, \bar{p}, \text{True}, \text{False}$ . The only possibility is that  $\varphi$  is equivalent to  $p$ , in which case  $\varphi \implies \exists \mathbf{F} p$ . Otherwise, if  $\varphi$  does not use a negation, then necessarily it uses the proposition  $p$ . Therefore,  $\varphi \in \{p, \exists \mathbf{F} p, \forall \mathbf{F} p, \exists \mathbf{G} p, \forall \mathbf{G} p, \exists \mathbf{X} p, \forall \mathbf{X} p\}$ .

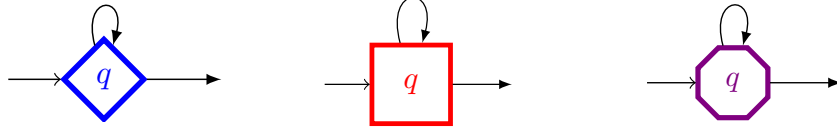


Figure 6: A Agent-1 state on the left, an Agent-2 state in the middle, and an Agent-3 state on the right.

One can then check that, in all these cases, we have  $\varphi \implies \exists \mathbf{F} p$ . Therefore, since  $K \models \varphi$ , we also have  $K \models \exists \mathbf{F} p$ . Hence,  $(p, \bar{p}, K)$  is a positive instance of 1-Reach.

The case  $\mathbf{H} = \mathbf{G}$  is dual: in that case, we consider the formula  $\varphi := \forall \mathbf{G} \bar{p}$ .  $\square$

With the two above lemmas, Theorem 79 follows.

*Proof.* The fact that the decision problem  $\text{CTL}_{\text{Learn}}(\mathbf{U}^t, \emptyset, \mathbf{B}^l, n)$  is in NL comes from Lemma 90. The fact that, if  $\mathbf{F} \in \mathbf{U}^t$  or  $\mathbf{G} \in \mathbf{U}^t$ , then it is also NL-hard comes from Lemma 94 and the fact that the reduction given in Definition 93 can be computed in logarithmic space.  $\square$

#### 4.4 ATL learning without the operator $\mathbf{X}$

We have seen in the previous section that CTL learning with the operator  $\mathbf{X}$  is NP-complete, while it can be solved in non-deterministic logarithmic space if this operator is not allowed anymore. In this section, we study the learning problem for ATL-formulas, that do not use the operator  $\mathbf{X}$ , and concurrent game structures, with two or three agents.

The cases of ATL learning with two or three agents are different. However, we start by giving some central definitions and establishing central lemmas that will be used in both cases for the NP-hardness proof.

##### 4.4.1 Alternating ATL-formulas and turn-based structures

We have introduced the notion of turn-based structures in Section 4.2.2. We will also use them in this subsection. As exemplified in Figure 6, whenever we draw turn-based game structures, we will use the following conventions:

- Blue diamond-shaped states are Agent-1 states;
- Red rectangle-shaped states are Agent-2 states;
- Violet octagon-shaped states are Agent-3 states.

Let us now introduce several useful definition on turn-based game structures. First of all, we consider self-looping turn-based structures.

**Definition 95.** Let  $\alpha \in \{2, 3\}$ ,  $\text{Ag} := [1 \dots, \alpha]$  and consider a  $(0, \emptyset)$ -proper  $\text{Ag}$ -turn-based structure  $T$ . This structure is self-looping if, for all states  $q$  in  $T$ , we have that  $q \in \text{Succ}(q)$ .

Furthermore, we will often use the states  $q^{\text{win}}$  and  $q^{\text{lose}}$  which will always satisfy the following:  $\text{Succ}(q^{\text{win}}) := \{q^{\text{win}}\}$ ,  $\text{Succ}(q^{\text{lose}}) := \{q^{\text{lose}}\}$ ,  $\pi(q^{\text{win}}) := \{p\}$ , and  $\pi(q^{\text{lose}}) := \{\bar{p}\}$ .

Such structures satisfy the lemma below.

**Lemma 96.** Let  $k \in \{2, 3\}$  and  $\text{Ag} := [1 \dots, k]$ . For all  $\text{ATL}^k$ -formulas  $\phi$ , the three ATL-formulas  $\phi$ ,  $\langle\langle \emptyset \rangle\rangle \mathbf{F} \phi$ , and  $\langle\langle \text{Ag} \rangle\rangle \mathbf{G} \phi$  are equivalent on all  $\text{Ag}$ -turn-based self-looping structures.



*Proof.* By definition of the operators  $\mathbf{F}$  and  $\mathbf{G}$ , we have  $\langle\langle \mathbf{Ag} \rangle\rangle \mathbf{G} \phi \implies \phi \implies \langle\langle \emptyset \rangle\rangle \mathbf{F} \phi$ . Furthermore, in any  $\mathbf{Ag}$ -turn-based structure, for any state  $q$  satisfying  $q \in \text{Succ}(q)$ , there is a strategy  $s$  for the coalition of agents  $\mathbf{Ag}$  such that  $\text{Out}^Q(q, s) = \{q^\omega\}$ . It follows that  $q \models \emptyset \mathbf{F} \phi$  implies  $q \models \phi$ , which itself implies  $q \models 1, 2 \mathbf{G} \phi$ .  $\square$

Let us now introduce below another notion on turn-based structures: (alternating) winning path. In a subsequent definition, we will also introduce the notion of alternating formulas and relate them with alternating winning paths.

**Definition 97.** Let  $\alpha \in \{2, 3\}$ ,  $\mathbf{Ag} := [1 \dots, \alpha]$  and consider a  $(0, \emptyset)$ -proper self-looping  $\mathbf{Ag}$ -turn-based structure  $T$ . Consider a state  $q \in Q$ . A winning path in  $T$  from  $q$  is a sequence of states  $\rho \in Q^{n+1}$ , for some  $n \in \mathbb{N}$ , such that:  $\rho[1] = q$ ,  $\pi(\rho[n+1]) = \{p\}$ , and for all  $1 \leq i \leq n$ , we have  $\pi(\rho[n+1]) = \{\bar{p}\}$  and  $\rho_{i+1} \in \text{Succ}(\rho_i)$ . We let  $\text{WinPath}_T(q)$  denote the set of all winning paths from  $q$ .

The winning path  $\rho \in \text{WinPath}_T(q)$  is safe if: for all  $1 \leq i \leq n$ , we have  $\text{Succ}(\rho[i]) = \{\rho[i], \rho[i+1]\}$  and  $\text{Succ}(\rho[n+1]) = \{\rho[n+1]\}$ . Note that, in that case, we have  $|\text{WinPath}_T(q)| = 1$ .

For any two coalitions  $A, A' \subseteq \mathbf{Ag}$  such that  $A \cap A' = \emptyset$ , we say that the winning path  $\rho \in \text{WinPath}_T(q)$  is  $(A, A', n)$ -alternating if: for all  $1 \leq i \leq n$ , we have  $\text{AgSt}(\rho[i]) \in A \cup A'$  and for all  $1 \leq i \leq n-1$ , we have  $\text{AgSt}(\rho[i]) \in A$  if and only if  $\text{AgSt}(\rho[i+1]) \in A'$ .

Let us now define the notion of alternating ATL-formulas.

**Definition 98.** Let  $\alpha \in \{2, 3\}$  and  $\mathbf{Ag} := [1 \dots, \alpha]$ . Consider two coalitions  $A, A' \subseteq \mathbf{Ag}$  such that  $A \cap A' = \emptyset$  (the union is disjoint). For all  $\mathbf{O} \subseteq \{\mathbf{F}, \mathbf{G}\}$ , we let  $\text{Op}_{A, A'}(\mathbf{O})$  denote the set  $\text{Op}_{A, A'}(\mathbf{O}) := \{\langle\langle B \rangle\rangle \mathbf{H} \mid B \cap A' = \emptyset \text{ or } B \cap A = \emptyset, \mathbf{H} \in \mathbf{O}\}$ . Then,  $\text{Op}_{A, A'}(\mathbf{O})$ -formulas refer to the set of  $\text{ATL}(\{p\}, \mathbf{O}, \emptyset, \emptyset, 0)$ -formulas  $\phi$  that only use operators in  $\text{Op}_{A, A'}(\mathbf{O})$ .

Then, we say that an  $\text{Op}_{A, A'}(\mathbf{O})$ -formula  $\phi$  is  $(A, A', n)$ -alternating if (recall Notation 56):

$$\phi \in \text{Op}_{A, A'}(\mathbf{O})^* \cdot \langle\langle B_1 \rangle\rangle \mathbf{F} \cdot \text{Op}_{A, A'}(\mathbf{O})^* \cdots \text{Op}_{A, A'}(\mathbf{O})^* \cdot \langle\langle B_n \rangle\rangle \mathbf{F} \cdot \text{Op}_{A, A'}(\mathbf{O})^* \cdot p$$

where, for all odd  $i \leq n$ , we have  $A' \cap B_i \neq \emptyset$ , and for all even  $i \leq n$ , we have  $A \cap B_i \neq \emptyset$ .

We state below a lemma relating alternating formulas and turn-based structures with alternating winning paths.

**Lemma 99.** Let  $k \in \{2, 3\}$  and  $\mathbf{Ag} := [1 \dots, k]$ . Consider a self-looping structure  $T$  and let  $n \in \mathbb{N}$ . We have, for all states  $q \in Q$ , and two non-empty coalitions  $A, A' \subseteq \mathbf{Ag}$  such that  $A \cap A' = \emptyset$ :

- a) If all winning paths  $\rho \in \text{WinPath}_T(q)$  are  $(A, A', n)$ -alternating, then, all  $\text{Op}_{A, A'}(\{\mathbf{F}, \mathbf{G}\})$ -formulas  $\phi$  that accept the state  $q$  are  $(A, A', n)$ -alternating formulas.
- b) If  $|A| = |A'| = 1$  and there is a safe winning path from  $q$  that is  $(A, A', n)$ -alternating, then any  $\text{Op}_{A, A'}(\{\mathbf{F}, \mathbf{G}\})$ -formula that is  $(A, A', n)$ -alternating accepts the state  $q$ .

*Proof.* Let us first argue the following: if an  $\text{Op}_{A, A'}(\{\mathbf{F}, \mathbf{G}\})$ -formula  $\phi$  accepts a state reachable from some state  $q' \in Q$ , then  $\text{WinPath}_T(q') \neq \emptyset$ . Indeed, if  $\text{WinPath}_T(q') = \emptyset$ , then all states  $q''$  reachable from  $q'$  are such that  $\pi(q'') = \{\bar{p}\}$ . Therefore, the  $\text{Op}_{A, A'}(\{\mathbf{F}, \mathbf{G}\})$ -formula  $\phi$  — that does not use negations and that is such that  $\text{Prop}(\phi) = p$  — does not accept any state reachable from  $q'$ .

Now, we prove both items of the lemma by induction on  $n \in \mathbb{N}$ . Item a) holds when  $n = 0$  since all  $\text{Op}_{A, A'}(\{\mathbf{F}, \mathbf{G}\})$ -formulas  $\phi$  are  $(A, A', 0)$ -alternating. Furthermore, if there is a safe winning path from  $q$  that is  $(A, A', n)$ -alternating, it means that  $\pi(q) = \{p\}$  and  $\text{Succ}(q) = \{q\}$ .

Thus, since any  $\text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})$ -formula  $\phi$  is such that  $\text{Prop}(\phi) = \{p\}$  and does not use any negations, Item b) follows.

Assume now that Items a) and b) hold for some  $n \in \mathbb{N}$ . Let us first consider Item a). Assume that all winning paths  $\rho \in \text{WinPath}_T(q)$  are  $(A, A', n+1)$ -alternating. Let us assume that  $\text{WinPath}_T(q) \neq \emptyset$ , otherwise no  $\text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})$ -formula accepts  $q$ . In particular, it must be that  $\text{AgSt}(q) \in A$  and  $\pi(q) = \{\bar{p}\}$ . Let us show by induction on  $\text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})$ -formulas  $\phi$  the following property  $\mathcal{P}(\phi)$ : if  $q \models \phi$ , then there is a state  $q' \in \text{Succ}(q) \setminus \{q\}$  and some ATL-formula  $\phi'$  such that  $q' \models \phi'$  and  $\phi \in \text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})^* \cdot \langle\langle B \rangle\rangle \mathbf{F} \cdot \phi'$  such that  $A \cap B \neq \emptyset$ . The property  $\mathcal{P}(p)$  holds since  $q \not\models p$ . Assume now that  $\mathcal{P}(\phi)$  holds for some  $\text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})$ -formula  $\phi$ . Let  $\psi := \mathbf{O} \cdot \phi$  for some  $\mathbf{O} \in \text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})$ . Assume that  $q \models \psi$ . Then, there are two cases:

- Assume that  $\mathbf{O} = \langle\langle B \rangle\rangle \mathbf{F}$  for some coalition  $B \subseteq \text{Ag}$ . If  $q \models \phi$ , we can deduce  $\mathcal{P}(\psi)$  from  $\mathcal{P}(\phi)$ . Assume now that it is not the case, i.e.  $q \not\models \phi$ . If  $\text{AgSt}(q) \notin B$ , for all strategies  $s_B$  of the coalition  $B$ , we have  $q^\omega \in \text{Out}^Q(q, s_B)$ . This is not possible since  $q \models \psi$  and  $q \not\models \phi$ . In fact,  $\text{AgSt}(q) \in B$ . Since  $\text{AgSt}(q) \in A$ , this implies  $A \cap B \neq \emptyset$ . Now, since  $q \models \psi$ , there is a strategy  $s_B$  for the coalition  $B$  such that, for all  $\rho \in \text{Out}^Q(q, s_B)$ , we have  $\rho \models \mathbf{F} \phi$ . Consider some  $\rho \in \text{Out}^Q(q, s_B)$  and let  $i \in \mathbb{N}_1$  be the least index such that  $\rho[i] \models \phi$ . We have  $\rho[i] \in \text{Succ}(q) \setminus \{q\}$ . Since there is some  $j \geq i$  such that  $\rho[j] \models \phi$  (and  $\rho[j]$  is reachable from  $\rho[i]$ ), then we have  $\text{WinPath}_T(\rho[i]) \neq \emptyset$ . Therefore, since all winning paths from  $q$  are  $(A, A', n+1)$ -alternating, it follows that  $\text{AgSt}(\rho[i]) \in A'$ . Since  $A \cap B \neq \emptyset$ , we have  $A' \cap B = \emptyset$  (since  $\langle\langle B \rangle\rangle \mathbf{F} \in \text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})$ ), and thus  $\text{AgSt}(\rho[i]) \notin B$ . Furthermore, since  $\rho \in \text{Out}^Q(q, s_B)$ , it follows that the strategy  $s_B$  chooses to go to  $\rho[i]$  after looping  $i-1$  times on  $q$ . Thus, since  $\rho[i] \in \text{Succ}(\rho[i])$ , we have  $q^{i-1} \cdot (\rho[i])^\omega \in \text{Out}^Q(q, s_B)$  and  $q^{i-1} \cdot (\rho[i])^\omega \models \mathbf{F} \phi$ . Hence, since  $q \not\models \phi$ , it follows that  $\rho[i] \models \phi$ . That is, the property  $\mathcal{P}(\psi)$  holds.
- Assume that  $\mathbf{O} = \langle\langle B \rangle\rangle \mathbf{G}$  for some coalition  $B \subseteq \text{Ag}$ . Then,  $\psi \implies \phi$ , thus  $q \models \phi$  and we can deduce  $\mathcal{P}(\psi)$  from  $\mathcal{P}(\phi)$ .

We deduce that  $\mathcal{P}(\phi)$  holds for all  $\text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})$ -formulas  $\phi$ . Therefore, for all  $\text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})$ -formulas  $\phi$  such that  $q \models \phi$ , we have that there is a state  $q' \in \text{Succ}(q) \setminus \{q\}$  and some ATL-formula  $\phi'$  such that  $q' \models \phi'$  and  $\phi \in \text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})^* \cdot \langle\langle B \rangle\rangle \mathbf{F} \cdot \phi'$  such that  $A \cap B \neq \emptyset$ . Since all the winning paths from  $q$  are  $(A, A', n)$ -alternating, it follows that all the winning paths from  $q'$  are  $(A', A, n)$ -alternating. Thus, by our induction hypothesis, we have that  $\phi'$  is an  $(A', A, n)$ -alternating formula, and therefore  $\phi$  is an  $(A, A', n+1)$ -alternating formula. Hence, Item a) also holds at index  $n+1$ .

Consider now Item b). Assume that  $|A| = |A'| = 1$  and that there is a safe winning path from  $q$  that is  $(A, A', n+1)$ -alternating. We let  $\rho \in \text{WinPath}_T(q)$ . Note that  $\rho[1] = q$  with  $\text{AgSt}(q) \in A$ . Consider any ATL-formula  $\phi$  that is  $(A', A, n)$ -alternating. Let  $\psi := \langle\langle B \rangle\rangle \mathbf{F} \cdot \phi$  for some coalition  $B$  such that  $A \cap B \neq \emptyset$ . Since  $|A| = 1$ , this implies  $\text{AgSt}(q) \in B$ . Let us show that  $q \models \psi$ . The coalition of agents  $B$  has a strategy  $s_B$  to ensure that, for all  $\rho' \in \text{Out}^Q(q, s_B)$ , we have  $\rho'[2] = \rho[2]$ . Therefore, since the winning path  $\rho[2:]$  is safe and is  $(A', A, n)$ -alternating, it follows by our induction hypothesis that  $\rho[2] \models \phi$ . Hence,  $q = \rho[1] \models \psi$ . Furthermore, for all  $2 \leq i \leq n+2$ , there is a safe winning path from  $\rho[i]$  that is  $(A, A', n+2-i)$ -alternating or  $(A', A, n+2-i)$ -alternating. Hence, since  $\psi$  is both  $(A, A', n+2-i)$ -alternating and  $(A', A, n+2-i)$ -alternating, by our induction hypothesis, we have  $\rho[i] \models \psi$ . Furthermore, since the winning path  $\rho \in \text{WinPath}_T(q)$  is safe, it follows that the set of states reachable from  $q$  in  $T$  is equal to  $\{\rho[i] \mid 1 \leq i \leq n+2\}$ . Hence, from all states  $q'$  reachable from  $q$ , we have  $q' \models \psi$ . We can then deduce that, for all  $\mathbf{O} \in \text{Op}_{A,A'}(\{\mathbf{F}, \mathbf{G}\})^*$ , we have  $q \models \mathbf{O} \cdot \psi$ . Therefore, Item b) also holds at index  $n+1$ . The lemma follows.  $\square$

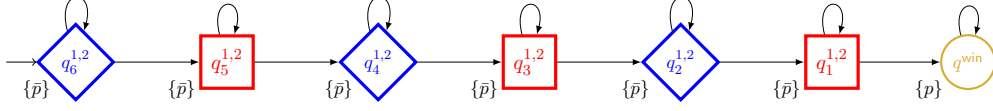


Figure 7: The turn based game structure  $T^{6:1,2}$ .

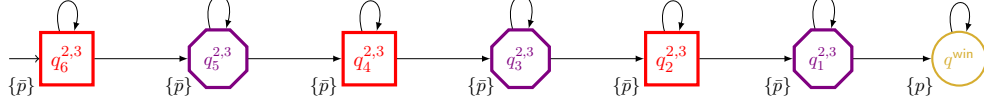


Figure 8: The turn based game structure  $T^{6:2,3}$ .

We conclude this section with a definition of alternating turn-based structures that we will use in the both proof of NP-hardness for ATL learning. We give the formal definition below, it is illustrated on Figures 7 and 8.

**Definition 100.** Let  $i \neq j \in \{1, 2, 3\}$ . We let:

- For all  $l \in \mathbb{N}_1$ ,  $Q^{l:i,j} := \{q_h^{i,j} \mid 1 \leq h \leq l\} \cup \{q^{win}\}$ ;
- For all  $h \in \mathbb{N}_1$ ,  $\text{AgSt}(q_{2h-1}^{i,j}) := j$  and  $\text{AgSt}(q_{2h}^{i,j}) := i$ ;
- For all  $h \in \mathbb{N}_1$ , we have:

$$\text{Succ}(q_h^{i,j}) := \begin{cases} \{q_h^{i,j}, q_{h-1}^{i,j}\} & \text{if } h > 1 \\ \{q_1^{i,j}, q^{win}\} & \text{if } h = 1 \end{cases}$$

Then, for all  $l \in \mathbb{N}_1$ , we define the turn-based structure  $T^{l:i,j} = \langle Q^{l:i,j}, I_{i,j,l}, 2, \{p\}, \pi, \text{AgSt}, \text{Succ} \rangle$  where  $I_{i,j,l} := \{q_1^{i,j}\}$ .

#### 4.4.2 ATL learning with two agents and operators $\mathbf{F}$ and $\mathbf{G}$

In this section, we focus on the case of ATL learning with two agents (Agent 1 and Agent 2). The goal of this subsection is to show the theorem below.

**Theorem 101.** Consider a set  $U^t \subseteq \text{Op}_{\text{Un}}$  of unary temporal operators such that  $\{\mathbf{F}, \mathbf{G}\} \subseteq U^t \subseteq \{\mathbf{F}, \mathbf{G}, \neg\}$ . Then, for all sets  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ , the decision problem  $\text{ATL}_{\text{Learn}}^2(U^t, \emptyset, B^l, n)$  is NP-complete.

**Overview of the reduction.** As for the CTL reduction, we follow the steps described in Section 4.2.1, with Step a) already taken care of in Section 4.2.2. Thus, we focus on  $\text{ATL}^2$ -formulas using only unary operators (and a single proposition). First, we define turn-based structures ensuring that: the proposition used is  $p$ , and the operators  $\langle\langle\{1, 2\}\rangle\rangle \mathbf{F}$ ,  $\langle\langle\emptyset\rangle\rangle \mathbf{G}$ ,  $\langle\langle\{2\}\rangle\rangle \mathbf{G}$  are not used. Since all the structures that we use are self-looping, Lemma 96 gives that the operators  $\langle\langle\{1, 2\}\rangle\rangle \mathbf{G}$ ,  $\langle\langle\emptyset\rangle\rangle \mathbf{F}$  are useless. Hence, we can focus on formulas using only the operators  $\langle\langle\{1\}\rangle\rangle \mathbf{F}$ ,  $\langle\langle\{2\}\rangle\rangle \mathbf{F}$ ,  $\langle\langle\{1\}\rangle\rangle \mathbf{G}$  and the proposition  $p$ , which are  $\text{Op}_{\{1\},\{2\}}(\{\mathbf{F}, \mathbf{G}\})$ -formulas.

Fix an instance  $(l, C, k)$  of the hitting set problem. We consider the bound  $B := 3l + 1 - k$ . Our idea is to focus on  $(\{1\}, \{2\}, 2l)$ -alternating formulas. To do so, we consider  $T^{2l:1,2}$  as positive structure and use Lemma 99 (Item a). Note that these  $(\{1\}, \{2\}, 2l)$ -alternating



Figure 9: The game  $T_p$  on the left and the game  $T_{\text{no } \emptyset \mathbf{G}, 2 \mathbf{G}}$  on the right.

formulas feature at least  $l$  occurrences of the operator  $\langle\langle\{1}\rangle\rangle \mathbf{F}$  and  $l$  occurrences of the operator  $\langle\langle\{2}\rangle\rangle \mathbf{F}$ . With the proposition occurring in the formula of size at most  $B$ , there remains  $l - k$  operators to use. In fact, we define a negative structure  $T_{\text{no } \langle\langle\{1}\rangle\rangle \mathbf{G}_{\geq k+1}}$  (see Figure 10) that is accepted by any  $\text{Op}_{\{1\},\{2\}}(\{\mathbf{F}, \mathbf{G}\})$ -formula featuring at least  $k+1$  sequences  $\langle\langle\{1}\rangle\rangle \mathbf{F} \cdot \langle\langle\{2}\rangle\rangle \mathbf{F}$  where the operators  $\langle\langle\{1}\rangle\rangle \mathbf{F}$  and  $\langle\langle\{2}\rangle\rangle \mathbf{F}$  are not separated by an operator  $\langle\langle\{1}\rangle\rangle \mathbf{G}$ . That way, we ensure that the remaining  $l - k$  operators are  $\langle\langle\{1}\rangle\rangle \mathbf{G}$  operators separating  $\langle\langle\{1}\rangle\rangle \mathbf{F}$  and  $\langle\langle\{2}\rangle\rangle \mathbf{F}$ . Hence, the formulas  $\phi^{\text{ATL}^{(2)}}(l, H)$  that we consider are  $(\{1\}, \{2\}, 2l)$ -alternating, features exactly  $l - k$  operators  $\langle\langle\{1}\rangle\rangle \mathbf{G}$ . The exact positions of these operators are given by the subset  $H \subseteq [1, \dots, l]$ . Note that, for this reduction, we use the fact that if there is a hitting set of size at most  $k$ , then there is one of size exactly  $k$ .

Then, there remains to define, given a subset  $C \subseteq [1, \dots, l]$ , a positive turn-based structure  $T_{l,C,2}$  such that  $\phi^{\text{ATL}^{(2)}}(l, H)$  accepts  $T_{l,C,2}$  if and only if  $H \cap C \neq \emptyset$ . The structure  $T_{l,C,2}$  (see Figure 11) is analogous to the structure  $T^{2l:1,2}$  except that the final state reached is  $q^{\text{lose}}$  instead of  $q^{\text{win}}$ . However, the Agent-1 states corresponding to the indices  $i \in C$  may not only continue towards the state  $q^{\text{lose}}$ , but also branch to a Agent-2 testing state that can branch to the losing state  $q^{\text{lose}}$ , or that can branch to the actual structure  $T^{2(i-1):1,2}$ . That way, these testing states are rejected by all  $\text{Op}_{\{1\},\{2\}}(\{\mathbf{F}, \mathbf{G}\})$ -formulas starting with the operator  $\langle\langle\{1}\rangle\rangle \mathbf{G}$ . Overall, we do obtain the desired equivalence.

**Formal definitions and proofs.** For readability, we will use the notations below.

**Notation 102.** For the coalition of agent  $A = \{i\}$  with  $i \in \{1, 2\}$ , the operators  $\langle\langle A \rangle\rangle \mathbf{F}$  and  $\langle\langle A \rangle\rangle \mathbf{G}$  will be denoted  $i \mathbf{F}$  and  $i \mathbf{G}$  respectively.

Now, the first step that we take is to define two simple turn-based game structures that will restrict the set of operators that we need to consider.

**Definition 103.** We define the trivial game structure  $T_p := \langle\{q^{\text{win}}\}, \{q^{\text{win}}\}, 2, \{p\}, \pi, \text{Ag}, \text{Succ}\rangle$ .

We also define the two-state turn-based game  $T_{\text{no } \emptyset \mathbf{G}, 2 \mathbf{G}} := \langle\{q, q^{\text{lose}}\}, \{q\}, 2, \{p, \bar{p}\}, \pi, \text{Ag}, \text{Succ}\rangle$  where  $\text{AgSt}(q) := 1$ ,  $\text{AgSt}(q^{\text{lose}}) := 1$ ,  $\text{Succ}(q) := \{q, q^{\text{lose}}\}$ ,  $\pi(q) := \{p\}$ .

These two games are depicted in Figure 9, and they satisfy the property below.

**Lemma 104.** Consider any ATL-formula  $\phi \in \text{ATL}^2(\text{Prop}_0, \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ . It accepts both  $T_p$  and  $T_{\text{no } \emptyset \mathbf{G}, 2 \mathbf{G}}$  if and only if  $\text{Prop}(\phi) = \{p\}$  and it does not use the operators  $\emptyset \mathbf{G}$  or  $2 \mathbf{G}$ .

*Proof.* If the proposition  $\text{Prop}(\phi) = \{\bar{p}\}$ , then  $\phi \not\models T_p$ . Hence,  $\text{Prop}(\phi) = \{p\}$  (note that  $|\text{Prop}(\phi)| = 1$  because  $\phi$  only uses unary operators).

Let us now show by induction the following property: an ATL-formula  $\phi \in \text{ATL}^2(\{p\}, \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$  accepts  $T_{\text{no } \emptyset \mathbf{G}, 2 \mathbf{G}}$  if and only if it does not use the operators  $\emptyset \mathbf{G}$  or  $2 \mathbf{G}$ . This holds straightforwardly for  $\phi = p$ . Assume now that it holds for some formula  $\phi \in \text{ATL}^2(\{p\}, \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$  accepts  $T_{\text{no } \emptyset \mathbf{G}, 2 \mathbf{G}}$ . Let  $\phi' = \langle\langle A \rangle\rangle \mathbf{H} \phi$  with  $A \subseteq \{1, 2\}$  and  $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$ .

- Assume that either  $\emptyset \mathbf{G}$  or  $2 \mathbf{G}$  occurs in  $\phi'$ . If  $\emptyset \mathbf{G}$  or  $2 \mathbf{G}$  occurs in  $\phi$ , then by our induction hypothesis, we have  $q \not\models \phi$ . Since we also have  $q^{\text{lose}} \not\models \phi$  (since  $\text{Prop}(\phi) = \{p\}$ )

it follows that  $q \not\models \phi'$ . Otherwise, we have  $\phi' = \langle\langle A \rangle\rangle \mathbf{G} \phi$  with  $1 \notin A$ . Since  $\text{AgSt}(q) = 1$ , for all strategies  $s$  for the coalition  $A$ , we have  $q \cdot (q^{\text{lose}})^\omega \in \text{Out}^Q(q, s)$  with  $q^{\text{lose}} \not\models \phi$ . Therefore,  $q \not\models \phi'$ .

- Assume that neither  $\emptyset \mathbf{G}$  nor  $2 \mathbf{G}$  occurs in  $\phi'$ . By our induction hypothesis, we have  $q \models \phi$ . Hence, if  $\mathbf{H} = \mathbf{F}$ , we have  $q \models \phi'$ . Otherwise,  $1 \in A$ . Since the Agent-1 strategy  $s_1$  that always loops on  $q$  is such that  $\text{Out}^Q(q, s) = \{q^\omega\}$ , it follows that  $q \models \phi'$ .

Thus, the property also holds for  $\phi'$ . In fact it holds for all formulas in  $\text{ATL}^2(\{p\}, \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ . The lemma follows.  $\square$

Furthermore, we have seen in Lemma 96 that the operators  $\emptyset \mathbf{F}$  and  $1, 2 \mathbf{G}$  are useless on self-looping structures. Thus, we restrict ourselves to promising formulas, i.e. formulas that only use the operators  $1 \mathbf{F}, 2 \mathbf{F}$  and  $1 \mathbf{G}$ . Note that we have not handled the operator  $1, 2 \mathbf{F}^6$  yet. It will be done on the fly later.

**Definition 105.** An  $\text{ATL}^2(\text{Prop}_0, \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$  formula is promising if  $\text{Prop}(\phi) = \{p\}$  and it only uses the operators  $1 \mathbf{F}, 2 \mathbf{F}$ , and  $1 \mathbf{G}$ .

Let us now consider the exact kinds of formulas that we consider, along with the structures that we define to encode the hitting set problem.

**Bounding the size of hitting sets** As mentioned at the beginning of this subsection, the way we encode a hitting set is by considering when the operator  $1 \mathbf{G}$  is used between the operators  $1 \mathbf{F}$  and  $2 \mathbf{F}$ . More precisely, we define the notion of concise formulas, a special kind of alternating formulas.

**Definition 106.** Let  $l \in \mathbb{N}_1$ . For all  $H \subseteq [1, \dots, l]$ , we denote by  $\phi^{\text{ATL}^2(2)}(l, H)$  the promising ATL-formula defined by:

$$\phi^{\text{ATL}^2(2)}(l, H) := 1 \mathbf{F} \text{Qt}_l 2 \mathbf{F} 1 \mathbf{F} \text{Qt}_{l-1} 2 \mathbf{F} \dots 1 \mathbf{F} \text{Qt}_1 2 \mathbf{F} p$$

where, for all  $i \in [1, \dots, l]$ , we have  $\text{Qt}_i \in \{\epsilon, 1 \mathbf{G}\}$  and  $\text{Qt}_i = \epsilon$  if and only if  $i \in H$ .

For all  $0 \leq k \leq l$ , we say that a promising ATL-formula  $\phi$  is  $(l, k)$ -concise if it is equal to  $\phi^{\text{ATL}^2(2)}(l, H)$  for some  $H \subseteq [1, \dots, l]$  with  $|H| = k$ . (In which case,  $|\phi| = 3l + 1 - k$ .)

Let us now define the turn-based structure that will force the use of a minimal number of  $1 \mathbf{G}$  operators, an example of which is depicted in Figure 10. This corresponds to the fact that, in the reduction, hitting sets have a bounded size.

**Definition 107.** Let  $i \in \{1, 2\}$ . For all  $k \in \mathbb{N}_1$ , we let  $T_{\text{no } 1 \mathbf{G} \geq k} := \langle Q_k^1 \mathbf{G}, I_k, 2, \{p\}, \pi, \text{Ag}, \text{Succ} \rangle$  where:

- $Q_k^1 \mathbf{G} := \{q_h^1 \mathbf{G} \mid 1 \leq h \leq 2k\} \cup \{q^{\text{win}}, q^{\text{lose}}\}$ ;
- $I_k := \{q_{2k}^1 \mathbf{G}\}$ ;
- For all  $1 \leq h \leq k$ ,  $\text{AgSt}(q_{2h-1}^1 \mathbf{G}) := 2$  and  $\text{AgSt}(q_{2h}^1 \mathbf{G}) := 1$ ;
- For all  $1 \leq h \leq k$ , we have:

$$\text{Succ}(q_{2h}^1 \mathbf{G}) := \{q_{2h}^1 \mathbf{G}, q_{2h-1}^1 \mathbf{G}\}$$

$$\text{Succ}(q_{2h-1}^1 \mathbf{G}) := \begin{cases} \{q_{2h-1}^1 \mathbf{G}, q_{2h-2}^1 \mathbf{G}, q^{\text{lose}}\} & \text{if } h > 1 \\ \{q_{2h-1}^1 \mathbf{G}, q^{\text{win}}, q^{\text{lose}}\} & \text{if } h = 1 \end{cases}$$

<sup>6</sup>That is, we have not shown that we can restrict ourselves to formulas that do not use this operator.

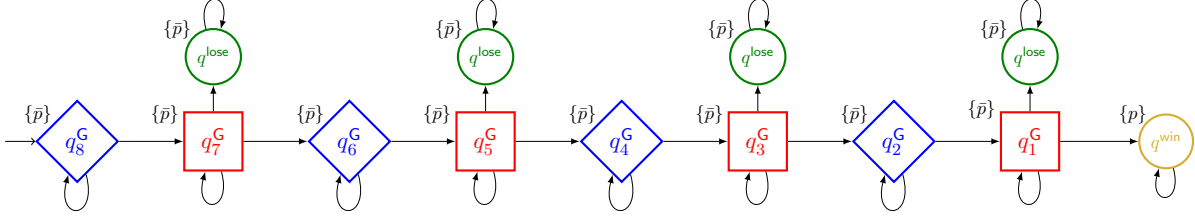


Figure 10: The turn-based game structure  $T_{\text{no } 1 \mathbf{G}_{\geq 4}}$ .

- For all  $q \in Q_k^1 \mathbf{G} \setminus \{q^{\text{win}}\}$ ,  $\pi(q) := \{\bar{p}\}$ .

Let us consider the lemma linking concise formulas and the turn-based structures  $T_{\text{no } 1 \mathbf{G}_{\geq k}}$ .

**Lemma 108.** *Let  $l \in \mathbb{N}_1$ . Consider a promising ATL-formula  $\phi$  that is  $(\{1\}, \{2\}, 2l)$ -alternating. Let  $k \leq l$ . If  $|\phi| \leq 3l + 1 - k$ , we have the following equivalence:*

$$q_{2(k+1)}^1 \mathbf{G} \not\models \phi \text{ if and only if } \phi \text{ is } (l, k)\text{-concise}$$

*Proof.* We prove by induction on  $n \in \mathbb{N}_1$  the property  $\mathcal{P}(n)$  stating the lemma for pairs  $(l, k)$  such that  $l \in \mathbb{N}_1$ ,  $0 \leq k \leq l$ , and  $l + k \leq n$ . We start with the case  $n = 0$ . The only possible pair is  $l = 1$ , and  $k = 0$ . Consider a promising ATL-formula  $\phi$  that is  $(\{1\}, \{2\}, 2)$ -alternating and such that  $|\phi| \leq 4$ . We have  $\phi = \mathbf{Qt}_1 \mathbf{1 F Qt}_2 \mathbf{2 F Qt}_3 p$ , with  $\mathbf{Qt}_1, \mathbf{Qt}_2, \mathbf{Qt}_3 \in \{\epsilon, \mathbf{1 F}, \mathbf{1 G}, \mathbf{2 F}\}$  and at most one of them not equal to  $\epsilon$ . One can then check that  $q_2^{\mathbf{G}} \not\models \phi$  if and only if  $\mathbf{Qt}_2 = \mathbf{1 G}$ , and thus  $\mathcal{P}(1)$  follows.

Consider now some  $n \in \mathbb{N}_1$  and assume that  $\mathcal{P}(n)$  holds. Let us show that  $\mathcal{P}(n + 1)$  holds. Consider any pair  $(l, k)$  such that  $l \in \mathbb{N}_1$ ,  $0 \leq k \leq l$ , and  $l + k = n + 1$ . Consider a promising ATL-formula  $\phi$  that is  $(\{1\}, \{2\}, 2l)$ -alternating and such that  $|\phi| \leq 3l + 1 - k$ . We let:

$$\phi = \mathbf{Qt}_1 \mathbf{1 F Qt}_2 \mathbf{2 F \phi'}$$

for some  $\mathbf{Qt}_1 \in (\mathbf{1 G}, \mathbf{2 F})^*$ ,  $\mathbf{Qt}_2 \in (\mathbf{1 F}, \mathbf{1 G})^*$ , and sub-formula  $\phi' \in \text{SubF}(\phi)$ . By definition,  $\phi'$  is  $(\{1\}, \{2\}, 2(l - 1))$ -alternating. First of all, note that, since  $\text{AgSt}(q_{2(k+2)}^{\mathbf{G}}) = 1$ , letting  $\psi := \mathbf{1 F Qt}_2 \mathbf{2 F \phi'}$ , we have that  $q_{2(k+1)}^{\mathbf{G}} \models \phi$  if and only if  $q_{2(k+1)}^{\mathbf{G}} \models \psi$ . There are two cases.

- Assume that  $\mathbf{Qt}_2 \in (\mathbf{1 F})^*$ . If  $q_{2(k+1)}^{\mathbf{G}} \not\models \psi$ , then we have  $k \geq 1$  (since  $q^{\text{win}} \models \phi'$ ) and it must be that  $q_{2k}^{\mathbf{G}} \not\models \phi'$ . Furthermore,  $\phi'$  is  $(\{1\}, \{2\}, 2(l - 1))$ -alternating, and  $|\phi'| \leq |\phi| - 2 \leq 3l + 1 - k = 3(l - 1) + 1 - (k - 1)$ . Therefore, by  $\mathcal{P}(n)$  (applied to the pair  $(l - 1, k - 1)$ ), we have that  $\phi'$  is  $(l - 1, k - 1)$ -concise, and of size  $|\phi'| = 3(l - 1) + 1 - (k - 1)$ . Since  $|\phi| \leq 3l + 1 - k$ , it follows that  $\mathbf{Qt}_1 = \mathbf{Qt}_2 = \epsilon$ , and  $\phi = \mathbf{1 F 2 F \phi'}$  is  $(l, k)$ -concise.

On the other hand, if  $\psi$  is  $(l, k)$ -concise, since  $\mathbf{Qt}_2 \neq \mathbf{1 G}$ , it follows that  $k \geq 1$ ,  $\phi = \mathbf{1 F 2 F \phi'}$  with  $\phi'$  a formula that is  $(l - 1, k - 1)$ -concise. Hence, by  $\mathcal{P}(n)$  (applied to the pair  $(l - 1, k - 1)$ ), we have  $q_{2k}^{\mathbf{G}} \not\models \phi'$ . This implies that all sub-formulas  $\phi'' \in \text{SubF}(\phi')$  are also such that  $q_{2k}^{\mathbf{G}} \not\models \phi''$  (because  $\text{AgSt}(q_{2k}^{\mathbf{G}}) = 1$ , and  $\phi' \in (\mathbf{1 F}, \mathbf{1 G}, \mathbf{2 F})^* \cdot \phi''$ ). It follows that it is also the case of all the sub-formulas of  $\phi$ . Therefore, since  $\pi(q_{2(k+1)}^{\mathbf{G}}) = \pi(q_{2k+1}^{\mathbf{G}}) = \{\bar{p}\}$ , and  $\text{Prop}(\phi) = \{p\}$ , we have  $q_{2(k+1)}^{\mathbf{G}} \not\models \phi$ .

- Assume now  $\mathbf{Qt}_2 \notin (\mathbf{1 F})^*$ . First of all, this implies  $k \leq l - 1$ . Indeed, if  $k = l$ , we have  $|\phi| \leq 2l + 1$ , and thus, since it is a  $(\{1\}, \{2\}, 2l)$ -alternating formula, it is equal to  $\phi = (\mathbf{1 F 2 F})^* p$ , and thus  $\mathbf{Qt}_2 = \epsilon$ . In fact, we do have  $k \leq l - 1$ . Furthermore, since  $\text{AgSt}(q_{2(k+1)}^{\mathbf{G}}) = 1$  and  $\text{AgSt}(q_{2k+1}^{\mathbf{G}}) = 2$  and  $q^{\text{lose}} \not\models \phi'$ , we have  $q_{2(k+1)}^{\mathbf{G}} \models \psi$  if and only if

$q_{2(k+1)}^{\mathbf{G}} \not\models \phi'$ . By  $\mathcal{P}(n)$  (applied to the pair  $(l-1, k)$ ), we have  $q_{2(k+1)}^{\mathbf{G}} \not\models \phi'$  if and only if  $\phi'$  is  $(l-1, k)$ -concise. Furthermore, if  $\phi'$  is  $(l-1, k)$ -concise, we have  $|\phi'| = 3(l-1) + 1 - k$ . Since  $|\phi| \leq 3l + 1 - k$  and  $\mathbf{Qt}_2 \notin (1\mathbf{F})^*$ , it follows that  $\phi = 1\mathbf{F}1\mathbf{G}2\mathbf{F}\phi'$ . In fact,  $\phi'$  is  $(l-1, k)$ -concise if and only if  $\phi$  is  $(l, k)$ -concise. Overall, we obtain that  $q_{2(k+1)}^{\mathbf{G}} \not\models \phi$  if and only if  $\phi$  is  $(l, k)$ -concise.

Hence, we have established  $\mathcal{P}(n+1)$ . In fact,  $\mathcal{P}(n)$  holds for all  $n \in \mathbb{N}$ , and the lemma follows.  $\square$

**Hitting sets should intersect the sets  $C_i$**  Let us now define the positive turn-based structures that encode the fact that the positions where there is a lack of operators  $1\mathbf{G}$  should match the subsets of integers  $C_i \subseteq [1, \dots, l]$  from the hitting set problem. We give the formal definition below, it is illustrated in Figure 11.

**Definition 109.** Let  $l \in \mathbb{N}_1$  and  $C \subseteq [1, \dots, l]$ . We let  $T_{l,C,2} := \langle Q^{l,C}, I_l, 2, \{p\}, \pi, \mathbf{Ag}, \mathbf{Succ} \rangle$  where (recall that the states  $q_h^{1,2}$  come from the turn-based structure  $T_{2l:1,2}$  from Definition 100):

- $Q^{l,C,2} := \{q_i \mid 1 \leq i \leq 2l\} \cup \{q_h^{1,2} \mid 1 \leq h \leq 2l\} \cup \{q_{2i-1}^{\mathbf{Test}} \mid i \in C\} \cup \{q^{\mathbf{lose}}, q^{\mathbf{win}}\}$ ;
- $I_l := \{q_{2l}\}$ ;
- For all  $1 \leq i \leq l$ ,  $\mathbf{AgSt}(q_{2i}) := 1$  and  $\mathbf{AgSt}(q_{2i-1}) := 2$ . For all  $i \in C$ , we have  $\mathbf{AgSt}(q_{2i-1}^{\mathbf{Test}}) := 2$
- For all  $1 \leq i \leq l$ , we have:

$$\mathbf{Succ}(q_{2i}) := \begin{cases} \{q_{2i}, q_{2i-1}\} & \text{if } i \notin C \\ \{q_{2i}, q_{2i-1}, q_{2i-1}^{\mathbf{Test}}\} & \text{if } i \in C \end{cases}$$

and

$$\mathbf{Succ}(q_{2i-1}) := \begin{cases} \{q_{2i-1}, q_{2(i-1)}\} & \text{if } i > 1 \\ \{q_{2i-1}, q^{\mathbf{lose}}\} & \text{if } i = 1 \end{cases}$$

and, for all  $i \in C$ :

$$\mathbf{Succ}(q_{2i-1}^{\mathbf{Test}}) := \begin{cases} \{q_{2i-1}^{\mathbf{Test}}, q^{\mathbf{lose}}, q_{2(i-1)}^{1,2}\} & \text{if } i > 1 \\ \{q_{2i-1}^{\mathbf{Test}}, q^{\mathbf{lose}}, q^{\mathbf{win}}\} & \text{if } i = 1 \end{cases}$$

- For all  $q \in Q^{l,C,2} \setminus \{q^{\mathbf{win}}\}$ ,  $\pi(q) := \{\bar{p}\}$ .

The above definition satisfies the lemma below.

**Lemma 110.** Consider any  $l \in \mathbb{N}_1$  and  $C, H \subseteq [1, \dots, l]$ . We have:

$$T_{l,C,2} \models \phi^{\mathbf{ATL}(2)}(l, H) \text{ if and only if } H \cap C \neq \emptyset$$

*Proof.* For all  $1 \leq i \leq l$ , we let  $H_i := H \cap [1, \dots, i]$ . Then, in the turn-based structure  $T_{l,C,2}$ , we prove by induction on  $1 \leq i \leq l$  the property  $\mathcal{P}(i)$ :  $q_{2i} \models \phi^{\mathbf{ATL}(2)}(i, H_i)$  if and only if  $H_i \cap C \neq \emptyset$ . We start with the case  $i = 1$ . There are two cases.

- Assume that  $H_1 \cap C = \{1\}$ . Then, we have  $\phi^{\mathbf{ATL}(2)}(1, H_1) = 1\mathbf{F}2\mathbf{F}p$ . Furthermore,  $q_1^{\mathbf{Test}} \in Q^{l,C,2}$  with  $q_1^{\mathbf{Test}} \models 2\mathbf{F}p$  since  $q^{\mathbf{win}} \in \mathbf{Succ}(q_1^{\mathbf{Test}})$ . Therefore,  $q_2 \models \phi^{\mathbf{ATL}(2)}(1, H_1)$  since  $q_1^{\mathbf{Test}} \in \mathbf{Succ}(q_2)$ .

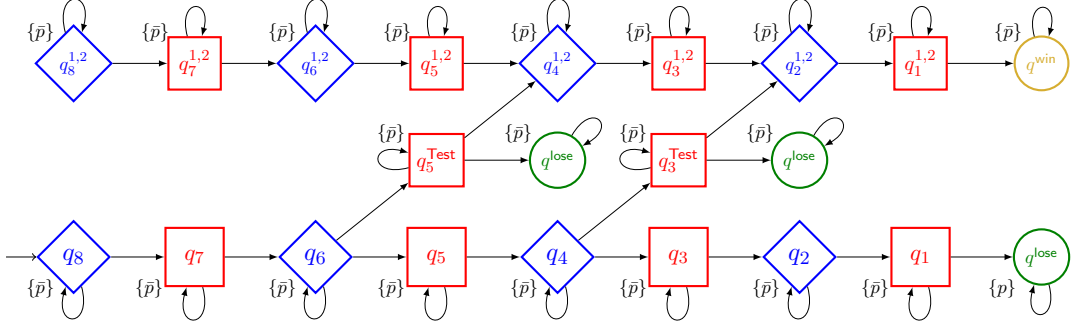


Figure 11: The turn-based game structure  $T_{4,\{2,3\},2}$ .

- Assume now that  $H_1 \cap C = \emptyset$ . If  $1 \in C$ , we have  $1 \notin H$ , thus  $\phi^{\text{ATL}(2)}(1, H_1) = 1 \mathbf{F} 1 \mathbf{G} 2 \mathbf{F} p$ . Since  $q^{\text{lose}} \in \text{Succ}(q_1^{\text{Test}})$ , we have  $q_1^{\text{Test}} \not\models 1 \mathbf{G} 2 \mathbf{F} p$  and  $q_1 \not\models 1 \mathbf{G} 2 \mathbf{F} p$ . Hence,  $q_2 \not\models \phi^{\text{ATL}(2)}(1, H_1)$ . On the other hand, if  $1 \notin C$ , there is no winning path from  $q_2$ , thus  $q_2 \not\models \phi^{\text{ATL}(2)}(1, H_1)$  by Lemma 99 (Item a).

Hence, the property  $\mathcal{P}(1)$  holds. Assume now that  $\mathcal{P}(i)$  holds for some  $1 \leq i \leq l-1$ . We have:

$$\phi^{\text{ATL}(2)}(i+1, H_{i+1}) = 1 \mathbf{F} \text{Qt}_{i+1} 2 \mathbf{F} \phi^{\text{ATL}(2)}(i, H_i)$$

with  $\text{Qt}_{i+1} = 1 \mathbf{G}$  if  $i+1 \notin H$  and  $\text{Qt}_{i+1} = \epsilon$  otherwise. As above, there are two cases.

- Assume that  $i+1 \in H \cap C$ . The only winning path from  $q_{2i}^{1,2}$  is  $(\{1\}, \{2\}, 2i)$ -alternating and safe. Therefore, by Lemma 99 (Item b), we have  $q_{2i}^{1,2} \models \phi^{\text{ATL}(2)}(i, H_i)$ . Hence,  $q_{2i+1}^{\text{Test}} \in Q^{l,C}$  with  $q_{2i+1}^{\text{Test}} \models 2 \mathbf{F} \phi^{\text{ATL}(2)}(i, H_i)$ . Therefore,  $q_{2(i+1)} \models \phi^{\text{ATL}(2)}(i+1, H_{i+1})$ .
- Assume now that  $i+1 \notin H \cap C$ . Let us show that  $q_{2(i+1)} \models \phi^{\text{ATL}(2)}(i+1, H_{i+1})$  if and only if  $q_{2i} \models \phi^{\text{ATL}(2)}(i, H_i)$ . First, if  $q_{2i} \models \phi^{\text{ATL}(2)}(i, H_i)$ , then  $q_{2i+1} \models 2 \mathbf{F} \phi^{\text{ATL}(2)}(i, H_i)$  and  $q_{2i+1} \models 1 \mathbf{G} 2 \mathbf{F} \phi^{\text{ATL}(2)}(i, H_i)$ . Thus, we have  $q_{2(i+1)} \models \phi^{\text{ATL}(2)}(i+1, H_{i+1})$ . Assume now that  $q_{2(i+1)} \models \phi^{\text{ATL}(2)}(i+1, H_{i+1})$ . Note that, the winning paths from  $q_{2(i+1)}$  are all  $(\{1\}, \{2\}, 2(i+1))$ -alternating, therefore, by Lemma 99 (Item a), no strict sub-formula of  $\phi^{\text{ATL}(2)}(i+1, H_{i+1})$  accept the state  $q_{2(i+1)}$ . Similarly, the winning paths from  $q_{2i+1}$  are all  $(\{2\}, \{1\}, 2i+1)$ -alternating, therefore, by Lemma 99 (Item a), no strict sub-formula of  $2 \mathbf{F} \phi^{\text{ATL}(2)}(i, H_i)$  accept the state  $q_{2i+1}$ . Then, there are two cases.
  - If  $i+1 \in C$ , we have  $i+1 \notin H$ , thus  $\phi^{\text{ATL}(2)}(i+1, H_{i+1}) = 1 \mathbf{F} 1 \mathbf{G} 2 \mathbf{F} \phi^{\text{ATL}(2)}(i, H_i)$ . Since  $q^{\text{lose}} \in \text{Succ}(q_{2i+1}^{\text{Test}})$ , we have  $\text{Succ}(q_{2i+1}^{\text{Test}}) \not\models 1 \mathbf{G} 2 \mathbf{F} \phi^{\text{ATL}(2)}(i, H_i)$ . Hence, with what we have argued above, we have  $q_{2i} \models \phi^{\text{ATL}(2)}(i, H_i)$ .
  - If  $i+1 \notin C$ , then  $\text{Succ}(q_{2(i+1)}) = \{q_{2(i+1)}, q_{2i+1}\}$ , hence as for the previous item, we have that  $q_{2i} \models \phi^{\text{ATL}(2)}(i, H_i)$ .

We have established that  $q_{2(i+1)} \models \phi^{\text{ATL}(2)}(i+1, H_{i+1})$  if and only if  $q_{2i} \models \phi^{\text{ATL}(2)}(i, H_i)$ . Furthermore, by  $\mathcal{P}(i)$ , we have  $q_{2i} \models \phi^{\text{ATL}(2)}(i, H_i)$  if and only if  $H_i \cap C \neq \emptyset$ . Since  $i+1 \notin H \cap C$ , it follows that  $H_{i+1} \cap C = H_i \cap C$ . Hence, we do obtain that  $q_{2(i+1)} \models \phi^{\text{ATL}(2)}(i+1, H_{i+1})$  if and only if  $H_{i+1} \cap C \neq \emptyset$ .

Hence,  $\mathcal{P}(i+1)$  holds. In fact,  $\mathcal{P}(i)$  holds for all  $1 \leq i \leq l$ . The lemma follows.  $\square$



**Definition of the reduction** We can finally define the reduction that we consider.

**Definition 111.** Consider an instance  $(l, C, k)$  of the hitting set problem  $\text{Hit}$ . We define:

- $\mathcal{P} := \{T_p, T_{\text{no } \emptyset \mathbf{G}, 2 \mathbf{G}}\} \cup \{T_{2l:1,2}\} \cup \{T_{(l, C_i, 2)} \mid 1 \leq i \leq n\}$ ;
- $\mathcal{N} := \{T_{\text{no } 1 \mathbf{G} \geq k+1}\} \cup \{T_{2l+1:2,1}\}$ .

Then, we define the input  $\text{In}_{(l, C, k)}^{\text{ATL}(2), \mathbf{F}, \mathbf{G}} := (\text{Prop}_0, \mathcal{P}, \mathcal{N}, 3l + 1 - k)$ .

This definition satisfies the lemma below.

**Lemma 112.** Let  $\mathbf{U}^t \subseteq \text{Op}_{\mathbf{U}^t}$  be a set of unary temporal operators such that  $\{\mathbf{F}, \mathbf{G}\} \subseteq \mathbf{U}^t \subseteq \{\mathbf{F}, \mathbf{G}, \neg\}$ . Consider an input  $(l, C, k)$  of the hitting set problem  $\text{Hit}$  and the corresponding input  $\text{In}_{(l, C, k)}^{\text{ATL}(2)}$ . Then,  $(l, C, k)$  is a positive instance of the decision problem  $\text{Hit}$  if and only if  $\text{In}_{(l, C, k)}^{\text{ATL}(2)}$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^2(\mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$  if and only if  $\text{In}_{(l, C, k)}^{\text{ATL}(2)}$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^2(\mathbf{U}^t, \emptyset, \emptyset, 0)$ .

*Proof.* Assume that  $(l, C, k)$  is a positive instance of the decision problem  $\text{Hit}$ . Consider a hitting set  $H \subseteq [1, \dots, l]$  of size at most  $k$ . We let  $H \subseteq H' \subseteq [1, \dots, l]$  be another hitting set of size exactly  $k$ . We let  $\phi := \phi^{\text{ATL}(2)}(l, H')$ . We have  $\phi \in \text{ATL}^2(\text{Prop}_0, \mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$  and  $|\phi| = 3l + 1 - k$ . Furthermore, since  $\text{Prop}(\phi) = \{p\}$  and  $\phi$  does not use the operators  $\emptyset \mathbf{G}, 2 \mathbf{G}$ , by Lemma 104, it accepts both structures  $T_p, T_{\text{no } \emptyset \mathbf{G}, 2 \mathbf{G}} \in \mathcal{P}$ . In addition,  $\phi$  is  $(\{1\}, \{2\}, 2l)$ -alternating, hence by Lemma 99 (Item b), it accepts  $T_{2l:1,2} \in \mathcal{P}$  (since there is a safe winning path from  $q_{2l}^{1,2}$  — in  $T_{2l:1,2}$  — that is  $(\{1\}, \{2\}, 2l)$ -alternating). On the other hand, all winning paths from  $q_{2l+1}^{2,1}$  are  $(\{2\}, \{1\}, 2l + 1)$ -alternating, while  $\phi$  is not  $(\{2\}, \{1\}, 2l + 1)$ -alternating, hence by Lemma 99 (Item a),  $\phi$  rejects the structure  $T_{2l+1:2,1} \in \mathcal{N}$ . In addition, by Lemma 108, since  $\phi$  is  $(l, k)$ -concise, it rejects  $T_{\text{no } 1 \mathbf{G} \geq k+1} \in \mathcal{N}$ . Finally, consider any  $1 \leq i \leq n$ . Since we have  $C_i \cap H' \neq \emptyset$ , it follows, by Lemma 110, that  $\phi$  accepts  $T_{(l, C_i, 2)} \in \mathcal{P}$ . Overall, the ATL-formula  $\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . Hence,  $\text{In}_{(l, C, k)}^{\text{ATL}(2)}$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^2(\mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 2)$ .

Furthermore, clearly, if  $\text{In}_{(l, C, k)}^{\text{ATL}(2)}$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^2(\mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$ , then it is also a positive instance of  $\text{In}_{(l, C, k)}^{\text{ATL}(2)}$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^2(\mathbf{U}^t, \emptyset, \emptyset, 0)$ .

Assume now that  $\text{In}_{(l, C, k)}^{\text{ATL}(2)}$  is a positive instance of the decision problem  $\text{ATL}_{\text{Learn}}^2(\mathbf{U}^t, \emptyset, \emptyset, 0)$ . Consider a formula  $\phi \in \text{ATL}^2(\text{Prop}_0, \mathbf{U}^t, \emptyset, \emptyset, 0)$  with  $|\phi| \leq 3l + 1 - k$  that accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . We let  $\phi = \text{Qt} \cdot x$  for some  $\text{Qt} \in (\text{Op}(2, \mathbf{U}^t))^*$  and  $x \in \text{Prop}_0$ . Let  $(\text{Qt}', x') := \text{UnNeg}(\text{Qt}, 0)$ . We let  $y \in \text{Prop}_0$  be such that  $y = x$  if and only if  $x' = 0$ . Finally, we let  $\phi' := \text{Qt}' \cdot y$ . By Lemma 61, we have:

- $|\text{Qt}'| \leq |\text{Qt}|$ , therefore  $|\phi'| \leq |\phi| \leq 3l + 1 - k$ ;
- $\text{Qt}' \in \text{Op}(2, \mathbf{U}^t \setminus \{\neg\})$ , therefore  $\phi' \in \text{ATL}^2(\text{Prop}_0, \mathbf{U}^t \setminus \{\neg\}, \emptyset, \emptyset, 0)$ ;
- Since all the structures in  $\mathcal{P}$  and  $\mathcal{N}$  are  $(0, \emptyset)$ -proper structures (and therefore, on those structures,  $p$  and  $\neg \bar{p}$  are equivalent),  $\phi'$  also accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ .

In addition, all the turn-based structures in  $\mathcal{P}$  and  $\mathcal{N}$  are self-looping, hence by Lemma 96, we may assume that neither of the operators  $\emptyset \mathbf{F}$  or  $1, 2 \mathbf{G}$  occur in  $\phi'$ . Furthermore, since  $\phi'$  accepts the games  $T_p \in \mathcal{P}$  and  $T_{\text{no } \emptyset \mathbf{G}, 2 \mathbf{G}} \in \mathcal{P}$ , by Lemma 104, we have  $\text{Prop}(\phi') = \{p\}$  and  $\phi'$  does not use the operators  $\emptyset \mathbf{G}, 2 \mathbf{G}$ . Furthermore, since  $\phi'$  rejects the game structure

$T_{2l+1:1,2} \in \mathcal{N}$ , it does not use the operator  $1, 2\mathbf{F}$  (since all sub-formulas of  $\phi'$  accept the state  $q^{\text{win}}$ ). In fact, the formula  $\phi'$  is promising.

Since  $\phi'$  accepts  $T_{2l:1,2} \in \mathcal{P}$ , and all the winning paths from the state  $q_{2l}^{1,2}$  are  $(\{1\}, \{2\}, 2l)$ -alternating, the formula  $\phi'$  is  $(\{1\}, \{2\}, 2l)$ -alternating by Lemma 99 (Item a). Furthermore, we have  $|\phi'| \leq 3l + 1 - k$ . Hence, since  $\phi'$  rejects  $T_{\text{no } 1\mathbf{G} \geq k+1} \in \mathcal{N}$ , by Lemma 108,  $\phi'$  is necessarily  $(l, k)$ -concise. We let  $H \subseteq [1, \dots, l]$  be such that  $|H| = k$  and  $\phi' = \phi^{\text{ATL}(2)}(l, H)$ . Then, consider some  $1 \leq i \leq n$ . Since  $\phi'$  accepts  $T_{(l, C_i, 2)} \in \mathcal{P}$ , it follows, by Lemma 110, that  $H \cap C_i \neq \emptyset$ . This holds for all  $1 \leq i \leq n$ . Therefore,  $H$  is a hitting set and  $(l, C, k)$  is a positive instance of the hitting set problem Hit.  $\square$

Theorem 101 follows.

*Proof.* This is direct consequence of Lemma 112, of the fact that the instance  $\text{In}_{(l, C, k)}^{\text{ATL}^2(\mathbf{X})}$  can be computed in logarithmic space from  $(l, C, k)$ , and of Theorem 65.  $\square$

We believe that the result of Theorem 101 would also hold with  $\mathbf{U}^t = \{\mathbf{F}, \neg\}$  or  $\mathbf{U}^t = \{\mathbf{G}, \neg\}$ , but handling these cases would induce many additional technical difficulties. In particular, Theorem 65 can be not applied as is.

#### 4.4.3 ATL learning with two agents and with one of the two operators $\mathbf{F}$ and $\mathbf{G}$

Let us now focus to the case of ATL learning with two agents where only one of the two operators  $\mathbf{F}$  or  $\mathbf{G}$  is allowed, without any negations. The goal of this subsection is to show the theorem below.

**Theorem 113.** *For all  $\mathbf{B}^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ , both decision problems  $\text{ATL}_{\text{Learn}}^2(\{\mathbf{F}\}, \emptyset, \mathbf{B}^l, n)$  and  $\text{ATL}_{\text{Learn}}^2(\{\mathbf{G}\}, \emptyset, \mathbf{B}^l, n)$  are P-complete.*

The idea is that with only the operator  $\mathbf{F}$  (it is the same with only the operator  $\mathbf{G}$ ), given a bound  $B \in \mathbb{N}_1$  and a set of propositions  $\text{Prop}$ , it is sufficient to consider only polynomially many ATL-formulas. The reason why comes from Lemma 80 (that we have stated when considering CTL-formulas). Indeed, this lemma states that, for all ATL-formulas  $\phi$ , for two coalitions of agents  $A_1, A_2$ , as soon as  $A_1 \subseteq A_2$  or  $A_2 \subseteq A_1$ , then  $\langle\langle A_1 \rangle\rangle \mathbf{F} \langle\langle A_2 \rangle\rangle \mathbf{F} \phi \equiv \langle\langle A \rangle\rangle \mathbf{F} \phi$ , with  $A \in \{A_1, A_2\}$  the largest of the two coalitions. However, since there are only two agents, there are only four coalitions:  $\emptyset, \{1\}, \{2\}, \{1, 2\}$ . Furthermore, we can make the following observations, for formulas without binary operators:

- If the operator  $1, 2\mathbf{F}$  is used, then all other  $\mathbf{F}$ -operators are useless;
- If at least one  $\mathbf{F}$ -operator is used, using additional  $\emptyset\mathbf{F}$  is useless;
- Using twice in a row the operator  $1\mathbf{F}$  or the operator  $2\mathbf{F}$  is useless.

This implies that it is sufficient to consider only polynomially many ATL-formulas. We define below the set of ATL-formulas to consider.

**Definition 114.** *Consider a set of propositions  $\text{Prop}$ . We define the sets  $\text{ATL}_{\mathbf{F}}(\text{Prop}, 2)$  and  $\text{ATL}_{\mathbf{G}}(\text{Prop}, 2)$  below. For  $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$ , we define:*

$$\text{Quant}_{\text{Alt}}^{\mathbf{F}} := \{\epsilon, \emptyset\mathbf{F}, 1, 2\mathbf{F}, (1\mathbf{F} \cdot 2\mathbf{F})^*, (1\mathbf{F} \cdot 2\mathbf{F})^* \cdot 1\mathbf{F}, (2\mathbf{F} \cdot 1\mathbf{F})^*, (2\mathbf{F} \cdot 1\mathbf{F})^* \cdot 2\mathbf{F}\}$$

and

$$\text{Quant}_{\text{Alt}}^{\mathbf{G}} := \{\epsilon, 1, 2\mathbf{G}, \emptyset\mathbf{G}, (1\mathbf{G} \cdot 2\mathbf{G})^*, (1\mathbf{G} \cdot 2\mathbf{G})^* \cdot 1\mathbf{G}, (2\mathbf{G} \cdot 1\mathbf{G})^*, (2\mathbf{G} \cdot 1\mathbf{G})^* \cdot 2\mathbf{G}\}$$

This definition satisfies the lemma below.

**Lemma 115.** *Consider any set of propositions  $\text{Prop}$ ,  $H \in \{\mathbf{F}, \mathbf{G}\}$ . For all sequences of quantifiers  $\text{Qt} \in \text{Op}(\{\mathbf{H}\}, 2)^*$ , there is some  $\text{Qt}' \in \text{Quant}_{\text{Alt}}^H$  such that:*

- $|\text{Qt}| \leq |\text{Qt}'|$ ;
- for all  $\text{ATL}^2$ -formulas  $\phi$ ,  $\text{Qt} \cdot \phi \equiv \text{Qt}' \cdot \phi$ .

*Proof.* We prove the result for  $H = \mathbf{F}$ , the case  $H = \mathbf{G}$  is analogous, and also relies entirely on Lemma 80. There are several cases:

- If  $\text{Qt} = \epsilon$ , then  $\text{Qt} \in \text{Quant}_{\text{Alt}}^H$ .
- Otherwise, assume that  $\text{Qt}$  only features the operator  $\emptyset \mathbf{F}$ . In that case, for all  $\text{ATL}^2$ -formulas  $\phi$ , we have  $\text{Qt} \cdot \phi \equiv \emptyset \mathbf{F} \cdot \phi$ .
- Otherwise, assume that  $\text{Qt}$  features the operator  $1, 2 \mathbf{F}$ . In that case, for all  $\text{ATL}^2$ -formulas  $\phi$ , we have  $\text{Qt} \cdot \phi \equiv 1, 2 \mathbf{F} \cdot \phi$ .
- Otherwise,  $\text{Qt}$  does not feature the operator  $1, 2 \mathbf{F}$  and features the operator  $1 \mathbf{F}$  or the operator  $2 \mathbf{F}$ . In that case, consider the sequence  $\text{Qt}'$  obtained from  $\text{Qt}$  by:
  - removing all  $\emptyset \mathbf{F}$  operators;
  - shrinking all sequences in  $(1 \mathbf{F})^+$  into  $1 \mathbf{F}$  and shrinking all sequences in  $(2 \mathbf{F})^+$  into  $2 \mathbf{F}$ .

By Lemma 80, for all  $\text{ATL}^2$ -formulas  $\phi$ , the formula  $\text{Qt} \cdot \phi$  is equivalent to  $\text{Qt}' \cdot \phi$ , and  $|\text{Qt}| \leq |\text{Qt}'|$ .

□

We handle in the definition below the case of formulas that may use a bounded amount of binary operators.

**Definition 116.** *Consider some  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and some  $H \in \{\mathbf{F}, \mathbf{G}\}$ . We define inductively on  $n \in \mathbb{N}$  the set  $\text{Rel}^{\text{ATL}^{(2)}}(B^l, H, n)$  as follows:*

- $\text{Rel}^{\text{ATL}^{(2)}}(B^l, H, 0) := \{\text{Qt} \cdot ? \mid \text{Qt} \in \text{Quant}_{\text{Alt}}^H\}$ ;
- For all  $n \in \mathbb{N}$ ,  $\text{Rel}^{\text{ATL}^{(2)}}(B^l, H, n+1) := \{\text{Qt} \cdot (\gamma_1 \bullet \gamma_2) \mid \text{Qt} \in \text{Quant}_{\text{Alt}}^H, \bullet \in B^l, \gamma_1, \gamma_2 \in \text{Rel}^{\text{ATL}^{(2)}}(B^l, H, n)\}$ .

*For all sets of propositions  $\text{Prop}$ ,  $n \in \mathbb{N}$ , and  $\gamma \in \text{Rel}^{\text{ATL}^{(2)}}(B^l, H, n)$ , we say that an  $\text{ATL}^2$ -formula  $\phi$  is  $\text{Prop}$ -from  $\gamma$  if it is equal to  $\gamma$ , up to replacing every  $?$  with some proposition in  $\text{Prop}$ .*

*For all bounds  $B \in \mathbb{N}$ , we let  $\text{Rel}^{\text{ATL}^{(2)}}(B^l, H, n, B)$  denote the set of elements in  $\text{Rel}^{\text{ATL}^{(2)}}(B^l, H, n)$  from which we can obtain a formula of size at most  $B$ .*

This definition satisfies the lemma below.

**Lemma 117.** *Consider some  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and some  $H \in \{\mathbf{F}, \mathbf{G}\}$ . For all  $n \in \mathbb{N}$ , there is a polynom  $q_n$  such that, for all  $B \in \mathbb{N}$ ,  $|\text{Rel}^{\text{ATL}^{(2)}}(B^l, H, n, B)| \leq q_n(n)$ .*

*Furthermore, for all non-empty sets of propositions  $\text{Prop}$ , for all  $\text{ATL}^2$ -formulas  $\phi$  of size at most  $B$ , there is some  $\gamma \in \text{Rel}^{\text{ATL}^{(2)}}(B^l, H, n, B)$  and a formula  $\phi'$  of size at most  $B$  that is  $\text{Prop}$ -obtained from  $\gamma$  and such that  $\phi \equiv \phi'$ .*

*Proof.* The first part of the lemma is a direct consequence of the fact that the number of sequences in  $\text{Quant}_{\text{Alt}}^H$  of size at most  $B \in \mathbb{N}$  is bounded by a polynomial in  $B$ .

The second part of the lemma is a direct consequence of Lemma 115.  $\square$

We deduce that the learning problem for ATL with two agents and only one of the operators  $\mathbf{F}, \mathbf{G}$  can be decided in polynomial time.

**Lemma 118.** *For all  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ , both decision problems  $\text{ATL}_{\text{Learn}}^2(\{\mathbf{F}\}, \emptyset, B^l, n)$  and  $\text{ATL}_{\text{Learn}}^2(\{\mathbf{G}\}, \emptyset, B^l, n)$  can be decided in polynomial time.*

*Proof.* Let  $H \in \{\mathbf{F}, \mathbf{G}\}$ ,  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ . Given an instance  $\text{In} = (\text{Prop}, \mathcal{P}, \mathcal{N}, B)$  of the decision problem  $\text{ATL}_{\text{Learn}}^2(\{H\}, \emptyset, B^l, n)$ , one can follow the following steps to check if  $\text{In}$  is a positive instance:

1. Loop over all elements  $\gamma \in \text{Rel}^{\text{ATL}(2)}(B^l, H, n, B)$ ;
2. Loop over all formulas  $\phi$  that can be Prop-obtained from  $\gamma$  of size at most  $B$ ;
3. Check whether or not this formula  $\phi$  accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ .

Following these steps is enough to decide if  $\text{In}$  is a positive instance of  $\text{ATL}_{\text{Learn}}^2(\{H\}, \emptyset, B^l, n)$  by Lemma 117. Furthermore, they can be executed in polynomial time. Indeed, by Lemma 117, the loop of step 1) is entered polynomially many times. Furthermore, all formulas with at most  $n$  occurrences of binary operators use at most  $2^n$  propositions. Hence, the loop of step 2) is entered at most  $|\text{Prop}|^{2^n}$  times, which is polynomial in Prop since  $n$  is fixed. Finally, the last step can be done in polynomial time as well. Thus, we do obtain a polynomial-time procedure deciding the problem  $\text{ATL}_{\text{Learn}}^2(\{H\}, \emptyset, B^l, n)$ .  $\square$

**P-hardness** As mentioned for the NL-hardness proof of the CTL learning problem without  $\mathbf{X}$ , to establish the P-hardness, we are going to exhibit a reduction from the reachability problem in two-player games  $\text{Reach}$ , introduced in Definition 91. We define the reduction that we consider. Note the three turn-based structures that we define below are depicted in Figure 12.

**Definition 119.** *Consider two propositions  $p, \bar{p}$  and a proper  $\{1, 2\}$ -turn-based structure  $T$  on  $\{p, \bar{p}\}$ . We let:*

- $T_{\bar{p}}$  be a trivial structure whose only state is labeled by the proposition  $\bar{p}$ ;
- $T_{\text{no } \mathbf{G}}$  be a two-state  $(0, \emptyset)$ -proper  $\{1, 2\}$ -turn-based structure whose only starting state, labeled by  $\bar{p}$ , belongs to Agent 1, with two outgoing edges, one that loops, and one that goes to a self-looping sink labeled by  $p$ ;
- $T_{\text{no } \langle\{2\}\rangle}$  be a two-state  $(0, \emptyset)$ -proper  $\{1, 2\}$ -turn-based structure similar to  $T_{\text{no } \mathbf{G}}$ , except that the starting state belongs to Agent 2, instead of Agent 1.
- $\mathcal{P} := \{T, T_{\text{no } \mathbf{G}}\}$  and  $\mathcal{N} := \{T_{\bar{p}}, T_{\text{no } \langle\{2\}\rangle}\}$ .

Then, we define the inputs  $\text{In}_{(p, \bar{p}, T)}^{\text{ATL}^2(2), \mathbf{F}} := (\{p, \bar{p}\}, \mathcal{P}, \mathcal{N}, 2)$  and  $\text{In}_{(p, \bar{p}, T)}^{\text{ATL}^2(2), \mathbf{G}} := (\{p, \bar{p}\}, \mathcal{N}, \mathcal{P}, 2)$ .

The definition above satisfies the lemma below.

**Lemma 120.** *Consider some set of unary operators  $U^t \subseteq \{\mathbf{X}, \mathbf{F}, \mathbf{G}, \neg\}$ , some set of binary operators  $B^l \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$ , some  $n \in \mathbb{N}$  and an input  $(p, \bar{p}, T)$  of the decision problem  $\text{Reach}$ . Let  $H \in \{\mathbf{F}, \mathbf{G}\}$ . If  $H \in U^t$ , the input  $(p, \bar{p}, T)$  is a positive instance of the  $\text{Reach}$  decision problem if and only if  $\text{In}_{(p, \bar{p}, T)}^{\text{ATL}^2(2), H}$  is a positive instance of the  $\text{ATL}_{\text{Learn}}^2(U^t, \emptyset, B^l, n)$  decision problem.*

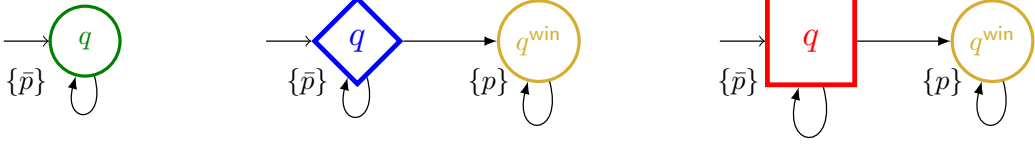


Figure 12: The game  $T_{\bar{p}}$  on the left, the game  $T_{\text{no } \mathbf{G}}$  in the middle and the game  $T_{\text{no } \langle\langle\{2}\rangle\rangle}$  on the right.

*Proof.* First assume that  $\mathbf{H} = \mathbf{F}$ . Assume that  $(p, \bar{p}, T)$  is a positive instance of **Reach**. We let  $\varphi := \langle\langle\{1}\rangle\rangle \mathbf{F} p$ . We have  $|\varphi| = 2$ . By assumption, we have  $T \models \varphi$ . In addition,  $T_{\text{no } \mathbf{G}} \models \varphi$ ,  $T_{\bar{p}} \not\models \varphi$  and  $T_{\text{no } \langle\langle\{2}\rangle\rangle} \not\models \varphi$ . Hence,  $\text{In}_{(p, \bar{p}, T)}^{\text{ATL}^2(2), \mathbf{F}}$  is a positive instance of the  $\text{ATL}_{\text{Learn}}^2(\mathbf{U}^t, \emptyset, \mathbf{B}^1, n)$  decision problem.

On the other hand, assume that  $\text{In}_{(p, \bar{p}, T)}^{\text{ATL}^2(2), \mathbf{F}}$  is a positive instance of the  $\text{ATL}_{\text{Learn}}^2(\mathbf{U}^t, \emptyset, \mathbf{B}^1, n)$  decision problem. Consider  $\phi$  a separating formula of size at most 2 that accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . Let us show that, on  $(0, \emptyset)$ -proper structures, we have  $\phi \implies \langle\langle\{1}\rangle\rangle \mathbf{F} p$ . If  $\phi$  is a proposition, uses a negation or a binary operator, then it is equivalent to either  $p, \bar{p}, \text{True}, \text{False}$  on  $(0, \emptyset)$ -proper structures. Since  $\phi$  accepts  $T_{\bar{p}}$  and rejects some negative structures, it necessarily is equivalent to  $p$ , and therefore it implies  $\langle\langle\{1}\rangle\rangle \mathbf{F} p$ . Assume now that  $\phi$  uses an operator in  $\{\mathbf{X}, \mathbf{F}, \mathbf{G}\}$  (in which we necessarily have  $\text{Prop}(\phi) = \{p\}$ ). Since  $\phi$  accepts the structure  $T_{\text{no } \mathbf{G}}$ , it does not use the operator  $\mathbf{G}$ . Therefore, if it does use an operator, it is not an  $\mathbf{G}$ -operator, and therefore the coalition of agents used cannot contain Agent 2 since  $\varphi$  rejects the structure  $T_{\text{no } \langle\langle\{2}\rangle\rangle}$ . Therefore, we have  $\varphi \in \{\langle\langle\{1}\rangle\rangle \mathbf{X} p, \langle\langle\{1}\rangle\rangle \mathbf{F} p, \langle\langle\emptyset\rangle\rangle \mathbf{X} p, \langle\langle\emptyset\rangle\rangle \mathbf{F} p\}$ . One can then check that, in all these cases, we have  $\varphi \implies \langle\langle\{1}\rangle\rangle \mathbf{F} p$ . Therefore, since  $T \models \varphi$ , we also have  $T \models \langle\langle\{1}\rangle\rangle \mathbf{F} p$ . Hence,  $(p, \bar{p}, T)$  is a positive instance of **Reach**.

The case  $\mathbf{H} = \mathbf{G}$  is dual: in that case, we consider the formula  $\varphi := \langle\langle\{2}\rangle\rangle \mathbf{G} \bar{p}$  (recall Proposition 59).  $\square$

The proof of Theorem 113 is now direct.

*Proof.* Let  $\mathbf{H} \in \{\mathbf{F}, \mathbf{G}\}$ . The decision problem  $\text{ATL}_{\text{Learn}}^2(\{\mathbf{H}\}, \emptyset, \mathbf{B}^1, n)$  can be decided in polynomial time by Lemma 118.

Furthermore, the decision problem  $\text{ATL}_{\text{Learn}}^2(\{\mathbf{H}\}, \emptyset, \mathbf{B}^1, n)$  is P-hard by Lemma 120 and the fact that the reduction given in Definition 119 can be computed in logarithmic space.  $\square$

#### 4.4.4 ATL learning with three agents

Let us now consider the case of ATL learning with three agents. The goal of this subsection is to show the lemma below.

**Theorem 121.** *For all sets  $\mathbf{B}^1 \subseteq \text{Op}_{\text{Bin}}^{\text{lg}}$  and  $n \in \mathbb{N}$ , both decision problems  $\text{ATL}_{\text{Learn}}^3(\{\mathbf{F}\}, \emptyset, \mathbf{B}^1, n)$  and  $\text{ATL}_{\text{Learn}}^3(\{\mathbf{G}\}, \emptyset, \mathbf{B}^1, n)$  are NP-complete.*

**Overview of the reduction.** We present the reduction with the operator  $\mathbf{F}$ . The reduction with the operator  $\mathbf{G}$  is the same, up to reversing the sets of positive and negative structures. First, as for the CTL and  $\text{ATL}^2$  reductions, we follow the steps described in Section 4.2.1, with Step a) already taken care of in Section 4.2.2. Thus, we focus on  $\text{ATL}^3$ -formulas using only the operator  $\mathbf{F}$  (and a single proposition). First, we define turn-based structures ensuring that: the proposition used is  $p$  (with a trivial positive structure), and the operators  $\langle\langle A \rangle\rangle \mathbf{F}$  such that  $1 \in A$  and  $\{2, 3\} \cap A \neq \emptyset$  are not used (with two negative structures). Furthermore, since all the structures that we use are self-looping, Lemma 96 gives

that the operator  $\langle\langle\emptyset\rangle\rangle \mathbf{F}$  is useless. Hence, we can focus on formulas using only the operators  $\langle\langle\{1}\rangle\rangle \mathbf{F}$ ,  $\langle\langle\{2}\rangle\rangle \mathbf{F}$ ,  $\langle\langle\{3}\rangle\rangle \mathbf{F}$ ,  $\langle\langle\{2,3}\rangle\rangle \mathbf{F}$  and the proposition  $p$ , which are  $\text{Op}_{\{1,\{2,3\}\}}(\{\mathbf{F}\})$ -formulas.

Fix an instance  $(l, C, k)$  of the hitting set problem. We consider the bound  $B := 2l + 1$ . Our idea is to focus on  $(\{1\}, \{2\}, 2l)$ -alternating formulas. As for the  $\text{ATL}^2$  case, we consider  $T^{2l:1,2}$  as positive structure and use Lemma 99 (Item a). Then, these  $(\{1\}, \{2\}, 2l)$ -alternating formulas feature at least  $l$  occurrences of the operator  $\langle\langle\{1}\rangle\rangle \mathbf{F}$  and  $l$  occurrences of operators  $\langle\langle A \rangle\rangle \mathbf{F}$  with  $2 \in A$ . Therefore the operator  $\langle\langle\{3}\rangle\rangle \mathbf{F}$  is not used.

Overall, the only thing that differs between the formulas that we consider is when the operators are  $\langle\langle\{2}\rangle\rangle \mathbf{F}$  and when the operators are  $\langle\langle\{2,3}\rangle\rangle \mathbf{F}$ . Thus, the formulas  $\phi^{\text{ATL}(3)}(l, H)$  that we consider are such that the set  $H$  entirely determines at which index  $i$  an operator  $\langle\langle\{2,3}\rangle\rangle \mathbf{F}$  occurs (when  $i \in H$ ), and at which index  $i$  an operator  $\langle\langle\{2}\rangle\rangle \mathbf{F}$  occurs (when  $i \notin H$ ). To ensure that  $|H| \leq k$ , we consider  $T^{2(k+1):1,3}$  as negative structure and we use Lemma 99 (Item b).

Then, there remains to define, given a subset  $C \subseteq [1, \dots, l]$ , a positive turn-based structure  $T_{l,C,3}$  such that  $\phi^{\text{ATL}(3)}(l, H)$  accepts  $T_{l,C,3}$  if and only if  $H \cap C \neq \emptyset$ . The structure  $T_{l,C,3}$  (see Figure 13) is similar to the structure  $T_{l,C,2}$ , except that the testing states are Agent-3 states.

**Formal definitions and proofs.** With the alternating turn-based structures that we have already defined (recall Definition 100) it is actually sufficient for the reduction define one additional type of turn-based structures to encode the fact that hitting set intersect all sets. Before we define it, let us first define the shape of the ATL-formulas that we will consider.

**Definition 122.** *Let  $l \in \mathbb{N}$ . For all  $H \subseteq [1, \dots, l]$ , an ATL-formula is an  $\phi^{\text{ATL}(3)}(l, H)$ -formula if:*

$$\phi = 1 \mathbf{F} \langle\langle A_l \rangle\rangle \mathbf{F} \dots 1 \mathbf{F} \langle\langle A_1 \rangle\rangle \mathbf{F} p$$

where for all  $i \in [1, \dots, l]$ , we have  $A_i \in \{\{2\}, \{2,3\}\}$  and  $A_i = \{2,3\}$  if and only if  $i \in H$ .

Let us now define the turn-based structure of interest whose definition is illustrated in Figure 13. This definition, and the subsequent lemma and proof are very similar to what we did with the turn-based structure  $T_{l,C,2}$  from Definition 109.

**Definition 123.** *Let  $l \in \mathbb{N}_1$  and  $C \subseteq [1, \dots, l]$ . We let  $T_{l,C,3} := \langle Q^{l,C,3}, I_{l,3}, 2, \{p\}, \pi, \text{AgSt}, \text{Succ} \rangle$  where (recall that the states  $q_h^{1,2}$  come from the turn-based structure  $T_{2l:1,2}$  from Definition 100):*

- $Q^{l,C,3} := \{q_i \mid 1 \leq i \leq 2l\} \cup \{q_h^{1,2} \mid 1 \leq h \leq 2l\} \cup \{q_{2i-1}^{\text{Test}} \mid i \in C\} \cup \{q^{\text{lose}}, q^{\text{win}}\};$
- $I_l := \{q_{2l}\};$
- For all  $2 \leq i \leq l + 1$ ,  $\text{AgSt}(q_{2i}) := 1$  and  $\text{AgSt}(q_{2i-1}) := 2$ . For all  $i \in C$ , we have  $\text{AgSt}(q_{2i-1}^{\text{Test}}) := 3$ .
- For all  $1 \leq i \leq l$ , we have:

$$\text{Succ}(q_{2i}) := \begin{cases} \{q_{2i}, q_{2i-1}\} & \text{if } i \notin C \\ \{q_{2i}, q_{2i-1}, q_{2i-1}^{\text{Test}}\} & \text{if } i \in C \end{cases}$$

and

$$\text{Succ}(q_{2i-1}) := \begin{cases} \{q_{2i-1}, q_{2(i-1)}\} & \text{if } i > 1 \\ \{q_{2i-1}, q^{\text{lose}}\} & \text{if } i = 1 \end{cases}$$

and, for all  $i \in C$ :

$$\text{Succ}(q_{2i-1}^{\text{Test}}) := \begin{cases} \{q_{2i-1}^{\text{Test}}, q_{2(i-1)}^{1,2}\} & \text{if } i > 1 \\ \{q_{2i-1}^{\text{Test}}, q^{\text{win}}\} & \text{if } i = 1 \end{cases}$$

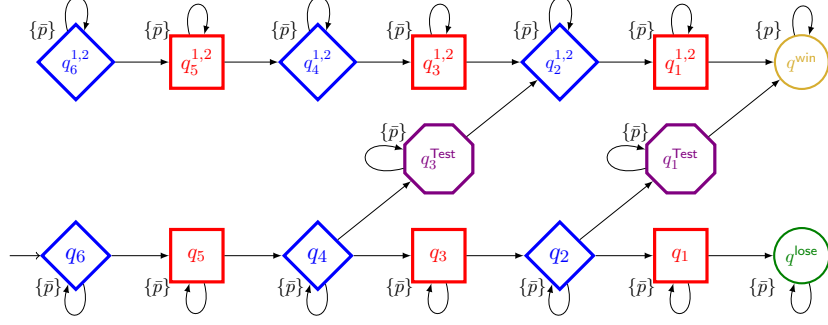


Figure 13: The turn-based game structure  $T_{3,\{1,2\},3}$ .

- For all  $q \in Q^{l,C,3} \setminus \{q^{\text{win}}\}$ , we have  $\pi(q) := \{\bar{p}\}$ .

The above definition satisfies the lemma below.

**Lemma 124.** Consider any  $l \in \mathbb{N}_1$  and  $C, H \subseteq [1, \dots, l]$  and an  $\phi^{\text{ATL}(3)}(l, H)$ -formula  $\phi$ . We have:

$$T_{l,C,3} \models \phi \text{ if and only if } H \cap C \neq \emptyset$$

*Proof.* For all  $1 \leq i \leq l$ , we let  $H_i := H \cap [1, \dots, i]$ . Then, in the turn-based structure  $T_{l,C,3}$ , we prove by induction on  $1 \leq i \leq l$  the property  $\mathcal{P}(i)$ :  $q_{2i} \models \phi^{\text{ATL}(3)}(i, H_i)$  if and only if  $H_i \cap C \neq \emptyset$ . We first handle the case  $i = 1$ . Consider the  $\phi^{\text{ATL}(3)}(1, H_1)$ -formula  $\phi = 1 \mathbf{F} \langle \langle A_1 \rangle \rangle \mathbf{F} p$ . There are two cases.

- Assume that  $H_1 \cap C = \{1\}$ . Then, we have  $A_1 = \{2, 3\}$ . Furthermore,  $q_1^{\text{Test}} \in Q^{l,C,3}$  with  $q_1^{\text{Test}} \models 2, 3 \mathbf{F} p$  since  $\text{AgSt}(q_1^{\text{Test}}) = 3$  and  $q^{\text{win}} \in \text{Succ}(q_1^{\text{Test}})$ . Thus,  $q_2 \models \phi$  since  $\text{AgSt}(q_2) = 1$ .
- Assume now that  $H_1 \cap C = \emptyset$ . If  $1 \in C$ , we have  $1 \notin H$ , thus  $A_1 = \{2\}$ . Since  $\text{AgSt}(q_1^{\text{Test}}) = 3$  and  $\pi(q_1^{\text{Test}}) = \{\bar{p}\}$ , we have  $q_1^{\text{Test}} \not\models \langle \langle A_1 \rangle \rangle \mathbf{F} p$ . Hence,  $q_2 \not\models \phi$ . On the other hand, if  $1 \notin C$ , we have  $\text{Succ}(q_2) = \{q_2, q_1\}$  and  $\text{Succ}(q_1) = \{q_1, q^{\text{lose}}\}$ . Therefore,  $q_2 \not\models \phi$ .

Hence, the property  $\mathcal{P}(1)$  holds. Assume now that  $\mathcal{P}(i)$  holds for some  $1 \leq i \leq l - 1$ . Consider the  $\phi^{\text{ATL}(3)}(i + 1, H_{i+1})$ -formula  $\phi$  defined by:

$$\phi := 1 \mathbf{F} \langle \langle A_{i+1} \rangle \rangle \mathbf{F} \phi'$$

with  $A_{i+1} = \{2, 3\}$  if  $i + 1 \in H$  and  $A_{i+1} = \{2\}$  otherwise, and  $\phi' = \phi^{\text{ATL}(3)}(i, H_i)$ . As above, there are two cases.

- Assume that  $i + 1 \in H \cap C$ . Since there is a safe winning path from  $q_{2i}^{1,2}$  that is  $(\{1\}, \{2\}, 2i)$ -alternating and the formula  $\phi'$  is  $(\{1\}, \{2\}, 2i)$ -alternating, it follows that  $q_{2i}^{1,2} \models \phi'$ , by Lemma 99 (Item b). Hence, we have  $q_{2i+1}^{\text{Test}} \in Q^{l,C,3}$  with  $q_{2i+1}^{\text{Test}} \models 2, 3 \mathbf{F} \phi'$ . Therefore,  $q_{2(i+1)} \models 1 \mathbf{F} \langle \langle A_{i+1} \rangle \rangle \mathbf{F} \phi'$  since  $\text{AgSt}(q_{2(i+1)}) = 1$ .
- Assume now that  $i + 1 \notin H \cap C$ . Let us show that  $q_{2(i+1)} \models \phi$  if and only if  $q_{2i} \models \phi'$ . First, if  $q_{2i} \models \phi'$ , then  $q_{2i+1} \models \langle \langle A_{i+1} \rangle \rangle \mathbf{F} \phi'$  since  $\text{AgSt}(q_{2i+1}) = 2 \in A_{i+1}$ . Thus, we have  $q_{2(i+1)} \models \phi$ . Assume now that  $q_{2(i+1)} \models \phi$ . Note that, the winning paths from  $q_{2(i+1)}$  are all  $(\{1\}, \{2, 3\}, 2(i + 1))$ -alternating, therefore, by Lemma 99 (Item a), no strict sub-formula of  $\phi$  accept the state  $q_{2(i+1)}$ . Similarly, the winning paths from  $q_{2i+1}$  are all  $(\{2, 3\}, \{1\}, 2i + 1)$ -alternating, therefore, by Lemma 99 (Item a), no strict sub-formula of  $\langle \langle A_{i+1} \rangle \rangle \mathbf{F} \phi^{\text{ATL}(2)}(i, H_i)$  accept the state  $q_{2i+1}$ . Then, there are two cases.

- If  $i + 1 \in C$ , we have  $i + 1 \notin H$ , thus  $\phi = \mathbf{1 F 2 F} \phi'$ . Furthermore, the winning paths from  $q_{2i}^{1,2}$  are all  $(\{1\}, \{2, 3\}, 2(i+1))$ -alternating, therefore, by Lemma 99 (Item a), no strict sub-formula of  $\phi'$  accept the state  $q_{2i}^{1,2}$ . Thus, since  $\text{AgSt}(q_{2i+1}^{\text{Test}}) = 3$ , it follows that  $q_{2i+1}^{\text{Test}} \not\models \mathbf{2 F} \phi'$ . Since  $\text{Succ}(q_{2(i+1)}) = \{q_{2(i+1)}, q_{2i+1}, q_{2i+1}^{\text{Test}}\}$  and  $\text{Succ}(q_{2i+1}) = \{q_{2i+1}, q_{2i}\}$ , we have  $q_{2i} \models \phi'$ .
- If  $i + 1 \notin C$ , then  $\text{Succ}(q_{2(i+1)}) = \{q_{2(i+1)}, q_{2i+1}\}$ , we have that  $q_{2i} \models \phi'$ .

We have established that  $q_{2(i+1)} \models \phi$  if and only if  $q_{2i} \models \phi'$ , with  $\phi' = \phi^{\text{ATL}(3)}(i, H_i)$ . Furthermore, by  $\mathcal{P}(i)$ , we have  $q_{2i} \models \phi'$  if and only if  $H_i \cap C \neq \emptyset$ . Since  $i + 1 \notin H \cap C$ , it follows that  $H_{i+1} \cap C = H_i \cap C$ . Hence, we do obtain that  $q_{2(i+1)} \models \phi$  if and only if  $H_{i+1} \cap C \neq \emptyset$ .

Hence,  $\mathcal{P}(i + 1)$  holds. In fact,  $\mathcal{P}(i)$  holds for all  $1 \leq i \leq l$ . The lemma follows.  $\square$

**Definition of the reduction** We can now define the reductions that we consider for the two cases  $\text{U}^t = \{\mathbf{F}\}$  and  $\text{U}^t = \{\mathbf{G}\}$ .

**Definition 125.** Consider an instance  $(l, C, k)$  of the hitting set problem Hit. We define:

- $\mathcal{P} := \{T_p, T_{2l:1,2}\} \cup \{T_{(l,C_i,3)} \mid 1 \leq i \leq n\}$ ;
- $\mathcal{N} := \{T_{2(l+1):1,2}, T_{2(k+1):1,3}\}$ .

Then, we define the inputs  $\text{In}_{(l,C,k)}^{\text{ATL}(3),\mathbf{F}} := (\text{Prop}_0, \mathcal{P}, \mathcal{N}, 2l+3)$  and  $\text{In}_{(l,C,k)}^{\text{ATL}(3),\mathbf{G}} := (\text{Prop}_0, \mathcal{N}, \mathcal{P}, 2l+3)$ .

This definition satisfies the lemma below.

**Lemma 126.** Let  $\text{H} \in \{\mathbf{F}, \mathbf{G}\}$ . An instance  $(l, C, k)$  of the hitting set problem is positive if and only if  $\text{In}_{(l,C,k)}^{\text{ATL}(3),\text{H}}$  is a positive instance of the  $\text{ATL}_{\text{Learn}}^3(\{\text{H}\}, \emptyset, \emptyset, 0)$  decision problem if and only if  $\text{In}_{(l,C,k)}^{\text{ATL}(3),\text{H}}$  is a positive instance of the  $\text{ATL}_{\text{Learn}}^3(\{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$  decision problem.

*Proof.* We consider the case where  $\text{H} = \mathbf{F}$ , the case  $\text{H} = \mathbf{G}$  is analogous.

Assume that  $(l, C, k)$  is a positive instance of the hitting set problem Hit. Consider a hitting set  $H \subseteq [1, \dots, l]$  and the  $\phi := \phi^{\text{ATL}(3),\mathbf{F}}(l, H) \in \text{ATL}^3(\{p\}, \{\mathbf{F}\}, \emptyset, \mathbf{B}^l, 0)$ . We have  $|\phi| = 2l + 1$ . Furthermore:

- the proposition used in  $\phi$  is  $p$ , therefore  $\phi$  accepts the structure  $T_p$ ;
- $\phi$  is  $(\{1\}, \{2\}, 2l)$ -alternating, thus it accepts the structure  $T_{2l:1,2}$  by Lemma 99 (Item b);
- for  $1 \leq i \leq n$ , we have  $C_i \cap H \neq \emptyset$ , hence by Lemma 124,  $\phi$  accepts the structure  $T_{(l,C_i,3)}$ ;
- $|H| \leq k$ , hence there are at most  $k$  times an operator  $\langle\langle A \rangle\rangle \mathbf{F}$  used in  $\phi$  with  $3 \in A$ . Hence,  $\phi$  is not  $(\{1\}, \{3\}, 2(k+1))$ -alternating. Thus, it rejects the structure  $T_{2(k+1):1,3}$  by Lemma 99 (Item a).

Therefore,  $\text{In}_{(l,C,k)}^{\text{ATL}(3),\mathbf{F}}$  is a positive instance of the  $\text{ATL}_{\text{Learn}}^3(\{\mathbf{F}\}, \emptyset, \emptyset, 0)$  decision problem.

Straightforwardly, if  $\text{In}_{(l,C,k)}^{\text{ATL}(3),\mathbf{F}}$  is a positive instance of  $\text{ATL}_{\text{Learn}}^3(\{\mathbf{F}\}, \emptyset, \emptyset, 0)$ , then it is also a positive instance of  $\text{ATL}_{\text{Learn}}^3(\{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ .

Assume now that  $\text{In}_{(l,C,k)}^{\text{ATL}(3),\mathbf{F}}$  is a positive instance of  $\text{ATL}_{\text{Learn}}^3(\{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ . Consider a formula  $\phi \in \text{ATL}^3(\text{Prop}_0, \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$  with  $|\phi| \leq 2l + 1$  that accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ . We let  $\phi = \text{Qt} \cdot x$  for some  $\text{Qt} \in (\text{Op}(3, \{\mathbf{F}, \mathbf{G}\}))^*$  and  $x \in \text{Prop}_0$ . Let  $(\text{Qt}', x') := \text{UnNeg}(\text{Qt}, 0)$ . We let  $y \in \text{Prop}_0$  be such that  $y = x$  if and only if  $x' = 0$ . Finally, we let  $\phi' := \text{Qt}' \cdot y$ . By Lemma 61, we have:



- $|\text{Qt}'| \leq |\text{Qt}|$ , therefore  $|\phi'| \leq |\phi| \leq 2l + 1$ ;
- $\text{Qt}' \in \text{Op}(3, \{\mathbf{F}, \mathbf{G}\})$ , therefore  $\phi' \in \text{ATL}^3(\text{Prop}_0, \{\mathbf{F}, \mathbf{G}\}, \emptyset, \emptyset, 0)$ ;
- Since all the structures in  $\mathcal{P}$  and  $\mathcal{N}$  are  $(0, \emptyset)$ -proper structures (and therefore, on those structures,  $p$  and  $\neg p$  are equivalent),  $\phi'$  also accepts  $\mathcal{P}$  and rejects  $\mathcal{N}$ .

The formula  $\phi'$  accepts the structure  $T_p$ , therefore  $\text{Prop}(\phi') = \{p\}$ . Furthermore, the formula  $\phi$  rejects both structures  $T_{2(l+1):1,2}$  and  $T_{2(k+1):1,3}$ . Hence, for all operators  $\langle\langle A \rangle\rangle \mathbf{F}$  used in  $\phi$ , if  $1 \in A$ , then  $\{2, 3\} \cap A = \emptyset$ . Therefore, since  $\phi$  accepts the structure  $T_{2l:1,2}$ , by Lemma 99 (Item a), the formula  $\phi$  is  $(\{1\}, \{2\}, 2l)$ -alternating. Since we have  $|\phi| \leq 2l + 1$ , this implies that

$$\phi = \langle\langle A_1^1 \rangle\rangle \mathbf{F} \langle\langle A_1^2 \rangle\rangle \mathbf{F} \dots \langle\langle A_l^1 \rangle\rangle \mathbf{F} \langle\langle A_l^2 \rangle\rangle \mathbf{F} p$$

where, for all  $1 \leq i \leq l$ , we have  $A_i^1 = \{1\}$  and  $2 \in A_i^2, 1 \notin A_i^2$ . However, since  $\phi$  rejects the structure  $T_{2(k+1):1,3}$ , by Lemma 99 (Item b), we have that  $\phi$  is not  $(\{1\}, \{3\}, 2(k+1))$ -alternating. Hence, there are at most  $k$  indices  $1 \leq i \leq l$  such that  $3 \in A_i^2$ . This implies that  $\phi = \phi^{\text{ATL}(3)}(l, H)$  for a set  $H \subseteq [1, \dots, l]$  such that  $|H| \leq k$ . Consider then any  $1 \leq i \leq n$ . Since the formula  $\phi$  accepts the structure  $T_{(l, C_i, 3)}$ , it follows by Lemma 124 that  $C_i \cap H \neq \emptyset$ . In fact,  $H$  is a hitting set and  $(l, C, k)$  is a positive instance of the hitting set problem Hit.

The case  $\mathbf{H} = \mathbf{G}$  is analogous. (It suffices to consider the negation of the separating formula.)  $\square$

The proof of Theorem 121 is now direct.

*Proof.* This is a direct consequence of Lemmas 126, the fact that the reductions from Definition 125 can be computed in logarithmic space and Theorem 65.  $\square$

## 5 Conclusion and future Work

In this work, we undertake an in-depth complexity analysis of the passive learning problems for LTL, CTL and ATL. Our results are gathered in Table 1, and could be roughly summarized as follows. When the number of occurrences of binary operators is unbounded, all the learning problems are NP-complete. On the other hand, when the number of occurrences of binary operators is bounded, discrepancies between the behaviors of LTL, CTL, and ATL learning appear: there are subsets of operators for which the learning problem is tractable with some number of agents, while it becomes untractable with more agents.

Overall, this paper essentially tackles reductions and hardness proofs, while the arguments that specific problems are in L, NL, P are (relatively) more straightforward. However, this is made possible by the fact that the bound in the size of the formula is given in unary. We have argued in Section 3.1.1 why we believe that it makes sense to consider such a setting. Nonetheless, the decision problems that would arise with a bound given in binary would certainly be interesting and challenging research questions (just like it was in [24]). Another interesting direction, which can be combined with the above one, could be, as is done in [24], to study the existence of tractable approximation algorithms.

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